# Riemann surfaces

Contents

## 1. Riemann surfaces. Introduction

### 1.1. **Basic definitions.**

**Definition 1.1.** A *Riemann surface* is a complex manifold of complex dimension 1.
More precisely: A Riemann surface $X$ is

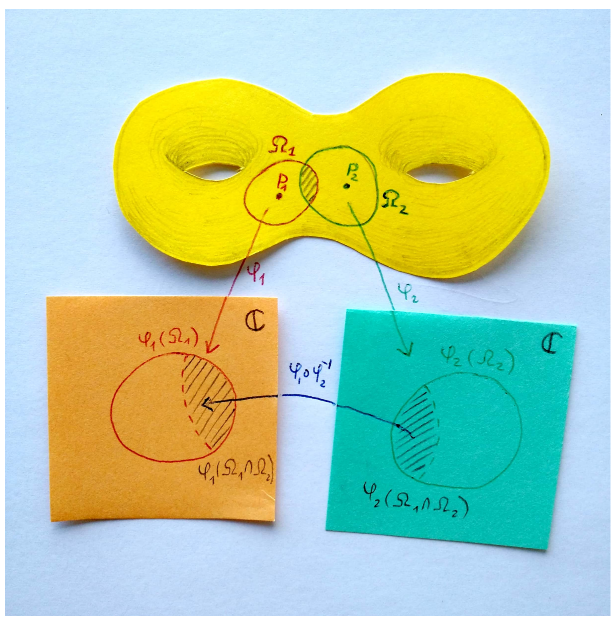
- Hausdorff space $X$
- any point $P \in X$ has an open neighbourhood $\Omega$ and a homeomorphism $\varphi : \Omega \to \varphi(\Omega) \subset \mathbb{C}$
- These homeomorphisms satisfy the consistency condition:

$$\varphi_1 \circ \varphi_2^{-1} \text{ is holomorphic on } \varphi_2(\Omega_1 \cap \Omega_2).$$

The maps $\phi_1 \circ \phi_2^{-1} : \phi_2(\Omega_1 \cap \Omega_2) \to \phi_1(\Omega_1 \cap \Omega_2)$ are called the *transition maps.*

This definition is illustrated in Figure 1.

Figure 1. Two holomorphic charts $(\Omega_1, \phi_1)$ ans $(\Omega_2, \phi_2)$ on a Riemann surface $X$ satisfy the consistency condition.




**Definition 1.2.** Such a pair $(\Omega, \varphi)$ is called a *holomorphic chart* on $X$ and the map $\varphi$ is called a local coordinate at $P$.

**Definition 1.3.** A collection of holomorhic charts $(\Omega_\alpha, \phi_\alpha)$ of $X$ with $X = \cup_\alpha \Omega_\alpha$ is an atlas of charts of $X$.

**Definition 1.4.** A complex valued function $\psi$ defined on an open set $U \subset X$ is *holomorphic* if for any $P \in U$ and any local coordinate $\phi$ at $P$, $\psi \circ \phi^{-1}$ is holomorphic near $z = \phi(P)$.

**Definition 1.5.** A continuous map $f : Y \to X$ between two Riemann surfaces is *holomorphic* if, for any holomorphic chart $\varphi : \Omega \to \mathbb{C}$ in $X$ the map $\varphi \circ f : f^{-1}(\Omega) \to \mathbb{C}$ is holomorphic.
Such a map is called an *isomorphism* of surfaces (or *biholomorphic* map), when it is a homeomorphism and its inverse is holomorphic.

**Definition 1.6.** We say that two Riemann surfaces $X$ and $Y$ are *equivalent* if there is a biholomorphic map $f : X \to Y$.

1.2. **Examples.**

- Any open domain in $\mathbb{C}$ defines a Riemann surface.
- Riemann sphere or projective line $\mathbb{CP}^1$. As a set we have

$$\mathbb{CP}^1 = \mathbb{C} \cup \infty.$$

  The holomorhpic structure on $\mathbb{P}^1\mathbb{C}$ is defined by the charts

$$\Omega_1 = \mathbb{CP}^1 \setminus \{\infty\} = \mathbb{C}, \quad \varphi_1(z) = z$$
$$\Omega_2 = \mathbb{CP}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}, \quad \varphi_2(z) = z^{-1} \text{ for } z \in \mathbb{C}^* \text{ and } \varphi_2(\infty) = 0.$$

  Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
- Consider $\mathbb{Z}$ as a subgroup of $\mathbb{C}$ under addition and form the quotient set $\mathbb{C}/\mathbb{Z}$. This has a standard quotient topology in which it is homeomorphic to a cylinder $S^1 \times \mathbb{R}$. We can make $\mathbb{C}/\mathbb{Z}$ into a Riemann surface in a following way. For each point $z \in \mathbb{C}$, we consider the disc $D_z$ centered on $z$ and with radius $1/4$. Clearly, the projection map $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ maps $D_z$ bijectively to the quotient space. Now we construct a chart about $\pi(z) \in \mathbb{C}/\mathbb{Z}$, taking $\Omega_z = \pi(D_z)$ and taking $\phi_z : \Omega_z \to \mathbb{C}$ as the local inverse of $\pi$. Then we cover $\mathbb{C}/\mathbb{Z}$ by some collection of charts of this form. The transition maps

$$\phi_{z_1} \circ \phi_{z_2}^{-1} : \phi_{z_2}(\Omega_{z_1} \cap \Omega_{z_2}) \to \phi_{z_1}(\Omega_{z_1} \cap \Omega_{z_2})$$

  are of the form

$$z \mapsto z + n$$

  for some $n \in \mathbb{Z}$, and these are clearly holomorphic.

**Definition 1.7.** A *meromorphic function* on a Riemann surface $X$ is a holomorphic map from $X$ to $\mathbb{CP}^1$.

**Exercise 1.** Prove that a meromorphic function on $\mathbb{CP}^1$ is rational.

**Exercise 2.** The Riemann surface $\mathbb{C}/\mathbb{Z}$ is biholomorphic to $\mathbb{C}\setminus\{0\}$.

## 2. Maps between Riemann surfaces. Basic properties

2.1. **Basic local properties of holomorphic maps.** First, we recall the following key fact about holomorphic functions. Let $U \subset \mathbb{C}$ be an open subset and let $f : U \to \mathbb{C}$ be a holomorphic function. Then for every $z_0 \in U$ the function $f$ has Taylor series expansion:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_k(z - z_0)^k + \ldots$$

and this series expansion converges is some open neighbourhood of $z$. More precisely, there exists a positive number $r$ such that the series converges at point $z$ if $|z - z_0| < r$. We use the existence of series expansion to prove the following two basic lemmas from complex analysis.

**Lemma 2.1.** *Let $f$ be a holomorphic function on an open neighbourhood $U$ of $0$ in $\mathbb{C}$ with $f(0) = 0$ and $f'(0) \neq 0$. Then there is another open neighbourhood $U' \subset U$ of $0$ such that $f$ is a homeomorphism from $U'$ to its image $f(U') \subset \mathbb{C}$ and the inverse map is also holomorphic. Moreover, we can choose the open neighbourhood $U'$ so that $f(U')$ is an open subset of $\mathbb{C}$.*

*Proof.* Function $f$ has Taylor expansion near the point 0

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_k(z - z_0)^k + \ldots, \qquad a_1 \neq 0.$$

This series has some convergence radius $r > 0$. Suppose $z_1, z_2 \in U$ and $|z_1|$, $|z_2| < r_0$ for some $r_0 \in (0, r)$. We have

$$f(z_1) - f(z_2) - a_1(z_1 - z_2) = a_2(z_1 - z_2)(z_1 + z_2) + a_3(z_1 - z_2)(z_1^2 + z_1 z_2 + z_2^2) + \ldots.$$

Therefore

$$(1) \qquad |f(z_1) - f(z_2) - a_1(z_1 - z_2)| \leq |z_1 - z_2| \left(|a_2| 2r_0 + |a_3| 3r_0^2 + \ldots + |a_k| 3r_0^{k-1} + \ldots\right).$$

We choose $r_0 \in (0, r)$ such that

$$|a_2| 2r_0 + |a_3| 3r_0^2 + \ldots + |a_k| 3r_0^{k-1} + \ldots < \frac{|a_1|}{2}.$$

Therefore, we deduce from equation (1)

$$|f(z_1) - f(z_2)| > \frac{|a_1|}{2} |z_1 - z_2|, \qquad z_1, z_2 \in U'.$$

Here we set $U' := \{z \in U \mid \ |z| < r_0\}$. Therefore the map $f : U' \to f(U')$ is invertible. We also see that from the above inequality that $f : U' \to f(U')$ is a homeomorphism.

Finally, we have to show that $f^{-1}$ is holomorphic. Let $w_1 \in f(U')$ and $z_1 := f^{-1}(w_1)$. Then

$$\lim_{\substack{w \to w_1 \\ w \in f(U')}} \frac{f^{-1}(w) - f^{-1}(w_1)}{w - w_1} = \lim_{z \to z_1 z \in U'} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)}.$$

This limit exists because $f$ is holomorphic on $U'$. Thus we have shown that $f^{-1}$ is holomorphic on $f(U')$.

Finally, if the derivative $f'(z)$ does not vanish for all points $z \in U'$, then the inverse mapping theorem in $\mathbb{R}^2$ implies that $f(U')$ is an open subset of $\mathbb{C}$. This finishes the proof of the lemma. ☐

**Lemma 2.2.** *Let $f$ be a holomorphic function on an open neighbourhood $U$ of $0$ in $\mathbb{C}$ with $f(0) = 0$ and $f \not\equiv 0$. Then there is a unique integer $k \geq 1$ such that on some smaller neighbourhood $U'$ of $0$, we can find a holomorphic function $g$ with $g'(0) \neq 0$ and $f(z) = g(z)^k$ on $U'$.*

**Definition 2.3.** A subset $S$ of a topological space $X$ is *discrete* if each point $P \in S$ has an open neighbourhood $U$ such that $U \cap S = \{P\}$.

**Theorem 2.4.** *(Uniqueness theorem) Let $X$, $Y$ be connected Riemann surfaces. Let $f : X \to Y$ and $g : X \to Y$ be holomorphic maps. Then either $f \equiv g$ or the set $\{p \in X | f(p) = g(p)\}$ is closed and discrete in $X$.*

*Proof.* The proof of this theorem follows from the uniqueness theorem for holomorphic maps on $\mathbb{C}$. It suffices to show that the following two sets of points on $X$

$$A = \{P \mid f \equiv g \text{ in a neighbourhood of } P\}$$
$$B = \{P \mid f \not\equiv g \text{ in a neighbourhood of } P\}$$

are open in $X$. □

**Proposition 2.5.** *Let $f : X \to Y$ be a holomorphic non-constant map between Riemann surfaces. Then for any $p \in X$ there is a unique integer $m = m_{p,f}$ such that there exist local parameters near $p$ and $f(p)$ such that both $p$ and $f(p)$ are mapped to 0 and $f$ has the form $z \mapsto z^m$.*

**Definition 2.6.** The integer $m$ given in the above proposition is the *multiplicity of $f$ at $p$*, denoted $\mathrm{mult}(f, p)$.

**Definition 2.7.** A point $p \in X$ is a *ramification point* for $f$ if $\mathrm{mult}(f, p) \geq 2$. A point $q \in Y$ is a *branch point* of $f$ if it is the image of a ramification point for $f$.

**Exercise 3.** Show that the set of ramification points of $f$ is discrete in $X$.

## 2.2. **Open mapping theorem.**

**Theorem 2.8.** *(Open mapping theorem) Let $f : X \to Y$ be holomorphic. If $f$ is non-constant, then $f$ maps open subsets in $X$ to open subsets in $Y$.*

We list below two corollaries of the open mapping theorem.

**Corollary 2.9.** *Let $X$ be compact connected, let $Y$ be connected, and let $f : X \to Y$ be holomorphic and non-constant. Then $f$ is surjective and $Y$ is compact.*

**Corollary 2.10.** *Let $X$ be a compact connected Riemann surface. Then*

- *every holomorphic map $f : X \to \mathbb{C}$ is constant*
- *every meromorphic function $f : X \to \mathbb{P}^1\mathbb{C}$ is surjective.*

## 2.3. **Local homeomorphisms and coverings.**

**Definition 2.11.** Let $M'$ and $M$ be manifolds. A map $\pi : M' \to M$ is said to be a *local homeomorphism* if each $x \in M'$ has a neighbourhood $U$ such that $\pi(U)$ is open in $M$ and $\pi|U$ is a homeomorphism (onto $\pi(U)$).

**Definition 2.12.** A local homeomorphism $\pi : M' \to M$ is called a *covering* if it is surjective and each $x \in M$ has a connected neighbourhood $V$ such that every connected component of $\pi^{-1}(V)$ is mapped by $\pi$ homeomorphically onto $V$.

*Remark.* In some sourses *coverings* are called *unramified* or *perfect coverings.*

**Lemma 2.13.** *Let $X$ and $Y$ be compact connected Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic map. Let $B \subset Y$ be the set of branch points of $f$. We denote by $Y'$ the set $Y \setminus B$ and by $X'$ the set $X \setminus f^{-1}(B)$. Then $f : X' \to Y'$ is a covering.*

### 2.4. **Degree of a holomorphic map.**

**Proposition 2.14.** *Let $f : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. For each $q \in Y$ define*

$$d_q(f) := \sum_{p \in f^{-1}(q)} \mathrm{mult}(f, p).$$

*Then $d_q(f)$ is constant (i. e. independent of $q$).*

**Definition 2.15.** Let $f : X \to Y$ be a non-constant map between compact connected Riemann surfaces. The *degree* of $f$, denoted $\deg(f)$, is the integer $d_q(f)$ for arbitrary $q \in Y$.

## 3. Constucting Riemann surfaces

3.1. **Quotients of Riemann surfaces.** Let $G$ be a group and let $X$ be a Riemann surface. Suppose that $G$ acts on $X$ by holomorphic transformations. More explicitly, we assume that there is a map

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying the conditions of the group action. Moreover, for a fixed $g \in G$ the map $x \mapsto g \cdot x$ is holomorphic on $X$. Here are several examples:

**Example 1**

Group $(\mathbb{Z}, +)$ acts on $\mathbb{C}$ by translations:

$$z \mapsto z + n, \quad n \in \mathbb{Z}.$$

**Example 2**

The cyclic group $(\mathbb{Z}/N\mathbb{Z}, +)$ acts on $\mathbb{C}^*$ by multiplication by roots of unity

$$z \mapsto e^{\frac{2\pi i k}{N}} z, \quad k \in \mathbb{Z}/N\mathbb{Z}.$$

**Example 3**

The group $\mathrm{PSL}_2(\mathbb{R}) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) | a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\} / \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ acts on the upper half-plane $\mathfrak{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ by linear-fractional transformations

$$\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) z := \frac{az + b}{cz + d}.$$

In this lecture we will study the quotients $X/G$ and answer the following question: Does $X/G$ have a natural structure of the Riemann surface?

First we address the general topological properties of $X/G$, in particular whether this quotient is a Hausdorff space.

**Definition 3.1.** A group action of a topological group $G$ on a topological space $X$ is said to be *a proper group action* if the mapping

$$G \times X \to X \times X$$
$$(g, x) \mapsto (gx, x)$$

is proper, i.e. pre-images of compact sets are compact.

**Lemma 3.2.** *Let $X$ be a Riemann surface. Suppose that a discrete group $\Gamma$ acts on $X$ and the action is proper. Then $X/\Gamma$ is a Hausdorff space.*

**Theorem 3.3.** *Let $X$ be a Riemann surface and let $\Gamma$ be a discrete group acting on $X$ such that:*

- *for any $\gamma \in \Gamma$ the map $x \mapsto \gamma \cdot x$ from $X$ to itself is holomorphic*
- *the action of $\Gamma$ on $X$ is proper*

*Then the quotient space $X/\Gamma$ possesses a natural structure of Riemann surface, characterized by the following property:*

*Let $\pi : X \to X/\Gamma$ denote the canonical map. For any open subset $U$ of $X/\Gamma$, $\pi^{-1}(U)$ is an open subset of $X$ and a function $f : U \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi : \pi^{-1}(U) \to \mathbb{C}$ is holomorphic.*

*Remarks:*

(1) The second condition implies that $X/\Gamma$ is a Hausdorff space.

(2) If $\Gamma$ acts freely on $X$ then the theorem is obvious.
(3) The quotient $X/\Gamma$ may be smooth even when some elements in $\Gamma \setminus \{0\}$ have fixed points. Consider the following example. Let $X$ be the unit disc $D = \{z \in \mathbb{C} | |z| < 1\}$ and let $\Gamma$ be the group $\mu_N$ of $N$-th roots of unity, acting by multiplication on $\mathbb{C}$. Then the $\Gamma$-invariant function on $D$ $z \mapsto z^N$ induces the isomorphism of Riemann surfaces

$$D/\mu_N \cong D.$$

The next lemma shows that all fixed points can be treated in a similar way.

**Lemma 3.4.** *Let $X$ be a Riemann surface and let $\Gamma$ be a discrete group acting on $X$ properly by biholomorphic transformations.*

a) *For any $P \in X$ its stabilizer in $\Gamma$*

$$\Gamma_P := \{\gamma \in \Gamma | \gamma \cdot P = P\}$$

*is a finite cyclic group.*

b) *There exists a chart $(U, z)$ of $X$ such that*

$$P \in U, \; z(P) = 0, \; z(U) = D$$
$$\gamma \in \Gamma_P \Longrightarrow \gamma \cdot U = U$$
$$\gamma \in \Gamma \setminus \Gamma_P \Longrightarrow \gamma \cdot U \cap U = \varnothing.$$

*Proof.* a) The group $\Gamma$ acts properly of $X$, thus for any compact subset $K$ of $X$, the set $\{\gamma \in \Gamma : \gamma(K) \cap K \neq \varnothing\}$ is finite. Thus for $P \in X$ we consider the set $K = \{P\}$ and obtain $\{\gamma \in \Gamma : \gamma(P) \cap P \neq \varnothing\}$ is finite. Since $\Gamma_P$ is a subset of $\{\gamma \in \Gamma : \gamma(P) \cap P \neq \varnothing\}$, the stabilizer $\Gamma_P$ is finite, too.

Next, we have to check whether $\Gamma_P$ is cyclic: We fix a local coordinate $z$ at $P$. For any $g \in \Gamma_P$ write

$$g(z) = \sum_{n=1}^{\infty} a_n(g) z^n;$$

this power series has no constant term since $g(P) = P$. Moreover, note that $a_1(g) \neq 0$, since $g$ is an automorphism of $X$ and hence has multiplicity one at every point, in particular at $P$.
Consider the function $a_1 : \Gamma_P \to \mathbb{C}^*$. Note that it is a homomorphism of groups: $a_1(gh)$ is calculated by computing the power series for $g(h(z))$, and this is

$$\begin{aligned} g(h(z)) &= g\left(\sum_{n=1}^{\infty} a_n(h) z^n\right) \\ &= \sum_{m=1}^{\infty} a_m(g) \left[\sum_{n=1}^{\infty} a_n(h) z^n\right]^m \\ &= a_1(g) a_1(h) z + \text{ higher order terms } . \end{aligned}$$

so that $a_1(gh) = a_1(g) a_1(h)$.

We check whether the homomorphism is bijective: let $g$ be a group element in the kernel of $a_1$. This means that $g(z) = z +$ higher order terms . In order to show that the kernel is trivial, we must show that in fact $g(z) = z$, i.e. that all higher order terms of $g$ are zero. Assume this is not the case: let $m \geq 2$ be the exponent of the first nonzero higher order term of $g$. Therefore $g(z) = z + az^m \mod z^{m+1}$ with $a \neq 0$.
Let $g^{[k]}$ be the $k$-times composition $g \circ \cdots \circ g$. By induction one can easily check that $g^{[k]}(z) =$

$z + kaz^m \mod z^{m+1}$. Since $\Gamma_P$ is finite, $g$ must have finite order. Hence for some $k$, $g^k$ is the identity, i.e. $g^{[k]}(z) = z$. Therefore for some $k$, $ka$ must be zero, forcing $a = 0$. This contradiction shows that in fact $g$ is the identity.
We have shown that $a_1$ is a bijective homomorphism. Since the only finite subgroups of $\mathbb{C}^*$ are cyclic, $\Gamma_P$ is a cyclic group.

b) Now as we know that $\Gamma_P$ is a finite cyclic group, we suppose that $\beta$ is a generator of $\Gamma_P$ and $N \in \mathbb{Z}_{\geq 1}$ is the order of $\beta$. Let $(\widetilde{U}, \widetilde{z})$ be a local chart around $P$ such that $\widetilde{z}(P) = 0$. By abuse of notation we consider $\beta$ as a map from $\widetilde{U}$ to $\mathbb{C}$ (in other words, we also denote by $\beta$ the function $\widetilde{z} \circ \beta \circ \widetilde{z}^{-1}$). Since the action of $\Gamma$ on $X$ is proper we can choose the neighbourhood $\widetilde{U}$ so that $\gamma \in \Gamma \setminus \Gamma_P$ implies $\gamma\widetilde{U} \cap \widetilde{U} = \varnothing$.

The set $U' := \cap_{k=0}^{N-1} \beta^{[k]}\widetilde{U}$ is an open neighbourhood of zero and $\beta(U') = U'$. Part a) of the lemma implies that $\beta'(0) = \zeta_N$, where $\zeta_N$ is a primitive $N$-th root of unity. Consider the map

$$g(\tilde{z}) := \frac{1}{N} \sum_{k=0}^{N-1} \zeta_N^{-k} \beta^{[k]}(\widetilde{z}), \quad \widetilde{z} \in U'. \tag{2}$$

Then $g'(0) = 1$ and therefore there is an open neighbourhood of zero $U'' \subset U'$ such that $g : U'' \mapsto g(U'')$ is a biholomorphism. Suppose that $r$ is a real number small enough such that the disk $D_r = \{z \mid |z| < r\}$ is contained in the open set $g(U'')$. We consider the following open subset of $U''$, the set $U := g^{-1}(D_r)$. The definition (5) implies that for $\widetilde{z} \in U'$

$$g(\beta(\widetilde{z})) = \zeta_N \, g(\widetilde{z}).$$

This identity immediately implies that $\beta(U) = U$ and therefore the set $U$ and the local coordinate $g \circ \widetilde{z}$ satisfy the conditions of the part b) of the lemma. This finishes the proof. ☐

**Exercise 4.** Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and

$$\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) / \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

via

$$z \mapsto \frac{az + b}{cz + d}.$$

1) Show that the set

$$\mathcal{F} := \{z \in \mathfrak{H} : \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}, \ |z| \geq 1\}$$

is the closure of a fundamental domain of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathfrak{H}$.
2) Show that the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathfrak{H}$ is proper.

One of the classical results in the theory of Riemann surfaces is the Uniformization theorem, which states that any connected Riemann surface may be obtained as a quotient from $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathfrak{H}$. If time permits, we will discuss this theorem later at the course.

3.2. **Algebraic curves as Riemann surfaces.** Let $P \in \mathbb{C}[X,Y]$ be an irreducible (non-constant) polynomial. Define the corresponding affine curve

$$C_P := \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0 \text{ and } (\frac{\partial P}{\partial X}, \frac{\partial P}{\partial Y})(x,y) \neq 0\}.$$

Our goal is to show that the affine curve $C_P$ has the structure of a Riemann surface.

**Theorem 3.5.** *Suppose that $(x_0, y_0)$ is a point in $C_P$ and $\frac{\partial P}{\partial y}(x_0, y_0) \neq 0$. Then there is a disc $D_1$ centered at $x_0$, a disc $D_2$ centered at $y_0$ and a holomorphic function $\phi : D_1 \to D_2$ with $\phi(x_0) = y_0$ such that*

$$C_P \cap D_1 \times D_2 = \{(z, \phi(z)) \mid z \in D_1\}.$$

*Proof.* The proof of this result can be found in [2][Theorem 1] □

Now we can define a holomorphic atlas on $C_P$. Let $(x_0, y_0)$ be a point on $C_P$. Then by definition of $C_P$ at least one of the partial derivatives $\frac{\partial P}{\partial X}$ or $\frac{\partial P}{\partial Y}$ does not vanish at this point. Suppose that $\frac{\partial P}{\partial x}(x_0, y_0) \neq 0$. Then we can apply Theorem 3.5. Consider the following open neighbourhood of $(x_0, y_0)$ in $C_P$, the set $U := C_P \cap (D_1 \times D_2)$. In this case the projection to the first coordinate $\pi_1 : U \to D_1$ defined by $(x, y) \mapsto x$ is a homeomorphism. We define $(U, \pi_1)$ to be a local coordinate around $(x_0, y_0)$. At the points where the partial derivative $\frac{\partial P}{\partial X}$ vanishes, we know that $\frac{\partial P}{\partial Y} \neq 0$ therefore by Theorem 3.5 and we can find a neighbourhood $\widetilde{U} = C_P \cap (\widetilde{D}_1 \times \widetilde{D}_2)$ such that the projection to the second coordinate is a homeomorphism. In this case we define $(\widetilde{U}, \pi_2)$ to be the holomorphic local chart. Around each point we can construct a holomorphic chart of the first kind or of the second kind (or both) and by Theorem 3.5 the transition maps are holomorphic. Also it is easy to see that this complex manifold is second countable.

Let us recall a definition from differential geometry:

**Definition 3.6.** Given a manifold $N$ of dimension $n$ we say a subset $M \subset N$ is an *$m$-dimensional complex submanifold* if around each point $p \in M$ there is a coordinate chart $(U, \phi)$ for $N$ in which $M$ looks like $\mathbb{C}^m \times 0 \subset \mathbb{C}^n$. That is, in such a preferred chart we have

$$\phi(M \cap U) = \phi(U) \cap (\mathbb{C}^m \times 0).$$

The affine curve $C_P$ is a one-dimensional complex submanifold of $\mathbb{C}^2$.

**Example 1** Consider a polynomial $P(X,Y) = X^2 - Y^3$. We have a biholomorphic map

$$\begin{aligned} \mathbb{C}^* &\to C_P \\ t &\mapsto (t^3, t^2) \end{aligned}$$

**Example 2** Hyperelliptic curve:
Let $Q \in \mathbb{C}[X]$ be a polynomial with simple roots. Then the polynomial $P(X,Y) = Y^2 - Q(X)$ is irreducible. Suppose that $\deg(Q) = 2g$. In the future lectures we will show that the curve $C_P$ has the topology of a compact surface of genus $g$ (sphere with $g$ handles) with two points deleted.
**Example 3** Affine Fermat curve:
Let $P(X,Y) = X^n + Y^n - 1$. In the future lectures we will show that the curve $C_P$ has the topology of a compact surface of genus $\frac{1}{2}(n-1)(n-2)$ with $n$ points deleted.

3.3. **Projective curves.** First let us define the projective space $\mathbb{PC}^n$. Consider the following equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$:

$$v \sim w \quad \text{if there exists } \lambda \in \mathbb{C}^* \text{ s.t. } v = \lambda w.$$

Now we define the projective space as

$$\mathbb{PC}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim .$$

We consider $\mathbb{P}^n\mathbb{C}$ as a set, as a topological space, and finally as a complex manifold.
Now we define a holomorphic atlas on $\mathbb{PC}^n$. Consider the following open sets $U_i \subset \mathbb{PC}^n$:

$$U_i := \mathbb{PC}^n \setminus \{\, [v_0 : \cdots : v_n] \in \mathbb{P}^n \mid v_i \neq 0 \,\}.$$

The set $U_i$ is biholomorphic to $\mathbb{C}^n$ via the map :

$$\phi_i : U_i \to \mathbb{C}^n, \quad [v_0 : \cdots : v_n] \mapsto \left( \frac{v_0}{v_i}, \ldots, \widehat{\frac{v_i}{v_i}}, \ldots, \frac{v_n}{v_i} \right).$$

Here the hat means the corresponding term is omitted.
What are the transition maps between $U_i$ and $U_j$?

$$\begin{aligned} &\text{We have } U_i \cap U_j = \{[v_1 : \cdots : v_{n+1}] \mid v_i \neq 0, v_j \neq 0\} \\ &\text{Define } V_i := \{(v_1, \ldots, v_{n+1}) \mid v_i \neq 0\}. \end{aligned}$$

$$\begin{array}{ccc} & U_i \cap U_j & \\ {}^{z_i}\swarrow & & \searrow^{z_j} \\ V_j \subset \mathbb{C}^n & \xrightarrow{\psi_{ij}} & V_i \subset \mathbb{C}^n \end{array}$$

$$(x_1, \ldots, x_n) \xrightarrow{z_i^{-1}} [x_1 : \cdots : 1 : \cdots : x_n] \xrightarrow{z_j} \left( \frac{x_1}{x_j}, \ldots, \frac{1}{x_j}, \ldots, \frac{x_n}{x_j} \right)$$

The $\frac{1}{x_j}$ is in the $i$-th position.
The transition map $\psi_{ij}$ is holomorphic.

**Lemma 3.7.** *Let $P \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial of degree $d > 0$.*

Then

$$\widetilde{C_P} = \left\{ [x:y:z] \in \mathbb{PC}^n \mid P(x,y,z) = 0, \left( \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right) |_{(x,y,z)} \neq 0 \right\}$$

is a smooth curve (submanifold of dimension 1) in $\mathbb{PC}^2$.

*Proof.* It suffices to show that $\overline{C_P} \cap U_i$ is a smooth curve for $i = 1, 2, 3$. Without loss of generality assume that $[x:y:z] \in U_3$ (i.e., $z \neq 0$).

By the results of section 3.2, it suffices to show that $\frac{\partial P}{\partial x}|_{(x,y,z)} \neq 0$ or $\frac{\partial P}{\partial y}\Big|_{(x,y,z)} \neq 0$.

We use the Eulers identity:

$$x \left.\frac{\partial P}{\partial x}\right|_{(x,y,z)} + y \left.\frac{\partial P}{\partial y}\right|_{(x,y,z)} + z \left.\frac{\partial P}{\partial z}\right|_{(x,y,z)} = \deg(P) \cdot P(x,y,z).$$

Since $(x,y,z) \in \overline{C_P}$ we have:

$$x \left.\frac{\partial P}{\partial x}\right|_{(x,y,z)} + y \left.\frac{\partial P}{\partial y}\right|_{(x,y,z)} + z \left.\frac{\partial P}{\partial z}\right|_{(x,y,z)} = 0$$

Suppose that $\frac{\partial P}{\partial x}|_{(x,y,z)} = \frac{\partial P}{\partial y}\Big|_{(x,y,z)} = 0$.

Then since $z \neq 0$ we also have

$$\left.\frac{\partial P}{\partial z}\right|_{(x,y,z)} = 0.$$

This contradicts the assumptions of the lemma. Therefore, at least one of the partial derivatives

$$\frac{\partial P}{\partial x} \text{ or } \frac{\partial P}{\partial y}$$

does not vanish at the point $(x,y,z)$.

This finishes the proof. ∎

**Example 1** Consider the polynomial $P(X,Y) = X^2 - Y^2 + 1$.

$$\widetilde{P}(X,Y,Z) = X^2 - Y^2 + Z^2$$

**Example 2**

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be distinct. We define

$$P(X,Y) = Y^2 - (X - z_1)(X - z_2)(X - z_3)(X - z_4).$$

$$C_P \subset \mathbb{C}^2$$

**Claim:**

$$\left( \frac{\partial P}{\partial X}, \frac{\partial P}{\partial Y} \right)\Bigg|_{(x,y)} \neq 0 \quad \text{for all } (x,y) \in \mathbb{C}^2 \text{ with } P(x,y) = 0.$$

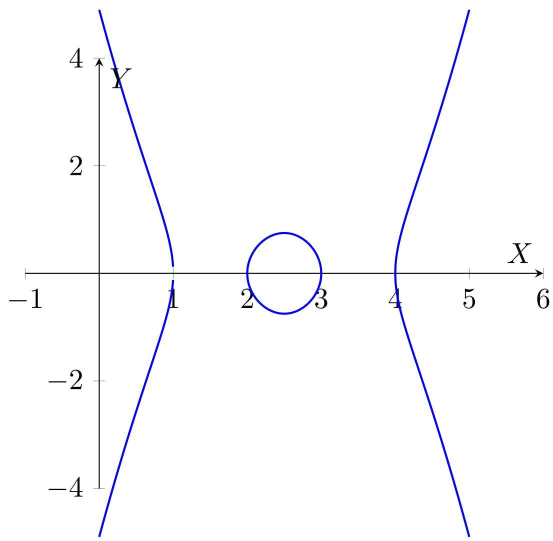
The claim implies that the curve $C_P$ is smooth.

The homogenized version of $P$ is:

$$\widetilde{P}(X,Y,Z) := Z^4 \cdot P\left( \frac{X}{Z}, \frac{Y}{Z} \right) = Y^2 Z^2 - (X - z_1 Z)(X - z_2 Z)(X - z_3 Z)(X - z_4 Z).$$

$$\overline{C}_P \subset \mathbb{P}^2.$$

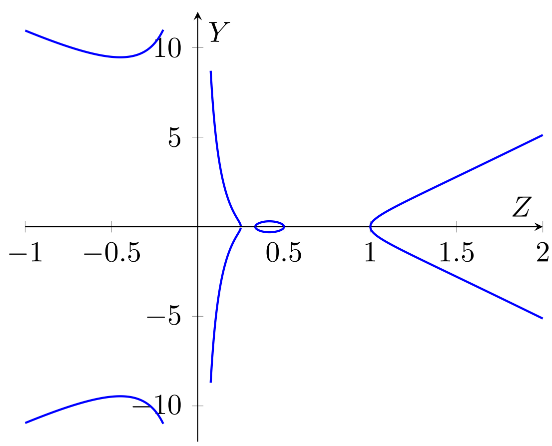
$\overline{C}_P \cap U_3$ is $C_P$.




Plot of $C_P$ for $z_1 = 1, z_2 = 2, z_3 = 3, z_4 = 4$.

$\overline{C}_P \cap U_1$ is an affine curve given by the equation:

$$P_1(Y, Z) := Y^2 Z^2 - (1 - z_1 Z)(1 - z_2 Z)(1 - z_3 Z)(1 - z_4 Z) = 0.$$



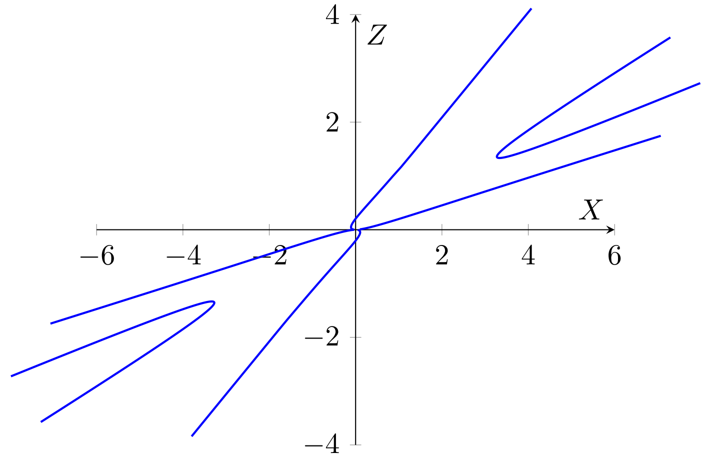

Plot of $\overline{C}_P \cap U_1$ for $z_1 = 1, z_2 = 2, z_3 = 3, z_4 = 4$.

$\overline{C}_P \cap U_2$ is an affine curve given by the equation:

$$P_2(X, Z) := Z^2 - (X - z_1 Z)(X - z_2 Z)(X - z_3 Z)(X - z_4 Z) = 0.$$

The poit $X = 0$, $Z = 0$: is a singular point on $C_{P_2}$.

$$\left.\frac{\partial P_2}{\partial X}\right|_{X=0, Z=0} = 0, \qquad \left.\frac{\partial P_2}{\partial Z}\right|_{X=0, Z=0} = 0.$$




Plot of $\overline{C}_P \cap U_2$ for $z_1 = 1, z_2 = 2, z_3 = 3, z_4 = 4$.

In this course we will prove that any compact Riemann surface is isomorphic to a projective algebraic curve.

## 4. Topology of Riemann surfaces

4.1. **Orientability and orientable manifolds.** In this section we define the orientability of differentiable manifolds.

**Definition 4.1.** Let $U$ be an open subset in $\mathbb{R}^n$. A differentiable function $f : U \to \mathbb{R}^d$ is *orientation preserving* if the Jacobian determinant of $f$ is positive at every point of $U$.

**Definition 4.2.** Let $M$ be an $n$-dimensional real differentiable manifold, that is all transition maps between charts of $M$ are differentiable. An *oriented atlas* on M is an atlas for which all transition functions are orientation preserving. M is *orientable* if it admits an oriented atlas.

**Exercise 5.** Show that a biholomorphic map defined on an open subset of $\mathbb{C}$ is orientation preserving. Here we assume that $\mathbb{C}$ is identified with $\mathbb{R}^2$ via the map $z \mapsto (\Re(z), \Im(z))$.

**Exercise 6.** Show that every Riemann surface is orientable.

*Remark.* The notion of orientability can be also defined for topological manifolds, however this definition goes beyond the scope of our course.

4.2. **Triangulation of compact Riemann surfaces.**

**Definition 4.3.** Let $X$ be a compact Riemann surface. A triangulation of $X$ consists of finitely many triangles $T_i$ such that $X = \cup_{i=1}^k T_i$. By a triangle we mean a closed subset of $X$ homeomorphic to a plane triangle, which is a compact subset of $\mathbb{C}$ bounded by three distinct straight lines. Furthermore, we require any two triangles $T_i$, $T_j$ be either pairwise disjoint, intersect at a single vertex, or intersect along a common edge.

**Theorem 4.4.** *Any compact Riemann surface can be triangulated.*

The proof of this theorem is non-trivial. It can be found in section 2.3 A of "Compact Riemann Surfaces, An Introduction to Contemporary Mathematics" by J. Jost.

4.3. **Planar model of a surface.**

**Definition 4.5.** A *planar diagram* is a polygon with $2n$ edges where pairs of edges are identified with either the same or opposite orientation. Given the quotient topology, the labelled edges are identified (i.e. glued together).

**Definition 4.6.** Let $\Pi$ be a planar diagram. Let $a_1, \dots a_{2n}$ be the edges of $\Pi$ written in the counter-clockwise order. Suppose that for $j > i$ $a_i$ and $a_j$ are glued. Then we set $a_j = a_i$ if they have the same orientation with respect to $\Pi$ and $a_j = a_i^{-1}$ if they have the opposite orientation. Now we define *the symbol* of the planar diagram $\Pi$ to be the product $a_1 \cdots a_{2n}$. Starting from a triangulation of a given surface, one can prove the following

**Theorem 4.7.** *Every compact, connected surface can be represented as a planar diagram.*

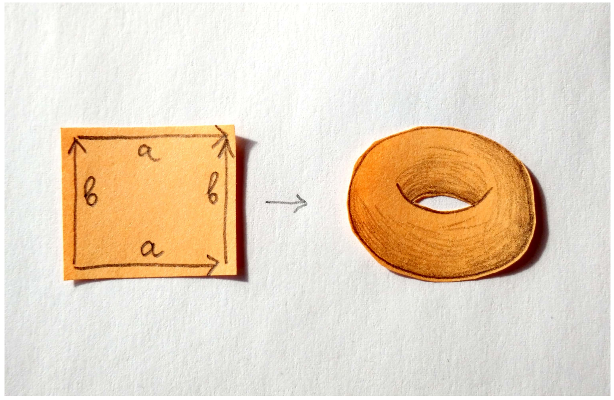



FIGURE 2. Planar diagram of a torus. The symbol of this diagram is $aba^{-1}b^{-1}$.

**Examples**

(1) Sphere is represented by a planar diagram $a\,a^{-1}$
(2) Real projective plane is defined as

$$\mathbb{P}^2\mathbb{R} = \mathbb{P}^2 := (\mathbb{R}^3 \setminus \{0\})/\sim$$

where the equivalence relation $\sim$ is given by $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for any $(x, y, z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. This is an example of a *non-orientable surface*. Real projective plane is represented by a planar diagram $a\,a$.

4.4. **Topological classification of compact surfaces.**

**Definition 4.8.** A connected sum is the new surface formed by combining two or more compact, connected surfaces in the following manner. Remove small discs from $S_1$, $S_2$, the two original surfaces, and glue the boundary circles together. (When dealing with triangulated surfaces, we can take the discs to be triangles.) We write $S_1 \# S_2$ for the connected sum of two surfaces.

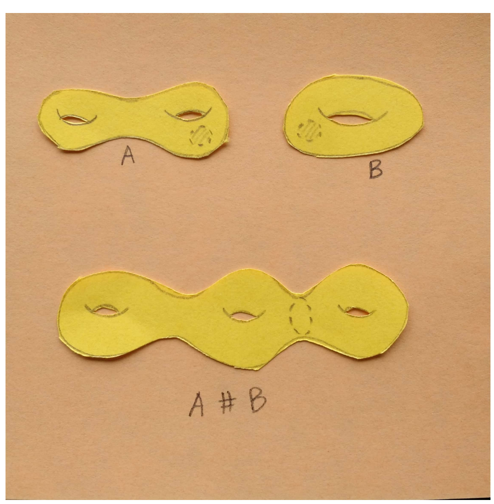



FIGURE 3. Connected sum of two surfaces.

**Theorem 4.9.** *(The classification theorem of closed surfaces) Any connected compact surface is homeomorphic either to a sphere, or to a connected sum of tori, or to a connected sum of real projective planes.*

In particular, for the orientable surfaces we have

**Theorem 4.10.** *Any orientable connected compact surface is homeomorphic to a sphere or a connected sum of $g$ tori for some $g \geq 0$.*

Let us briefly explain the idea of the proof:

**Step 1** let $S$ be a compact connected oriented surface. We show that $S$ has a planar model such that only edges with opposite directions are identified.
**Step 2** We show that $S$ is isomorphic to a surface given by a planar diagram $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ for some $g \geq 0$.
**Step 3** We show the surface represented by the planar diagram $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ is homeomorphic to the connected sum of $g$ tori.

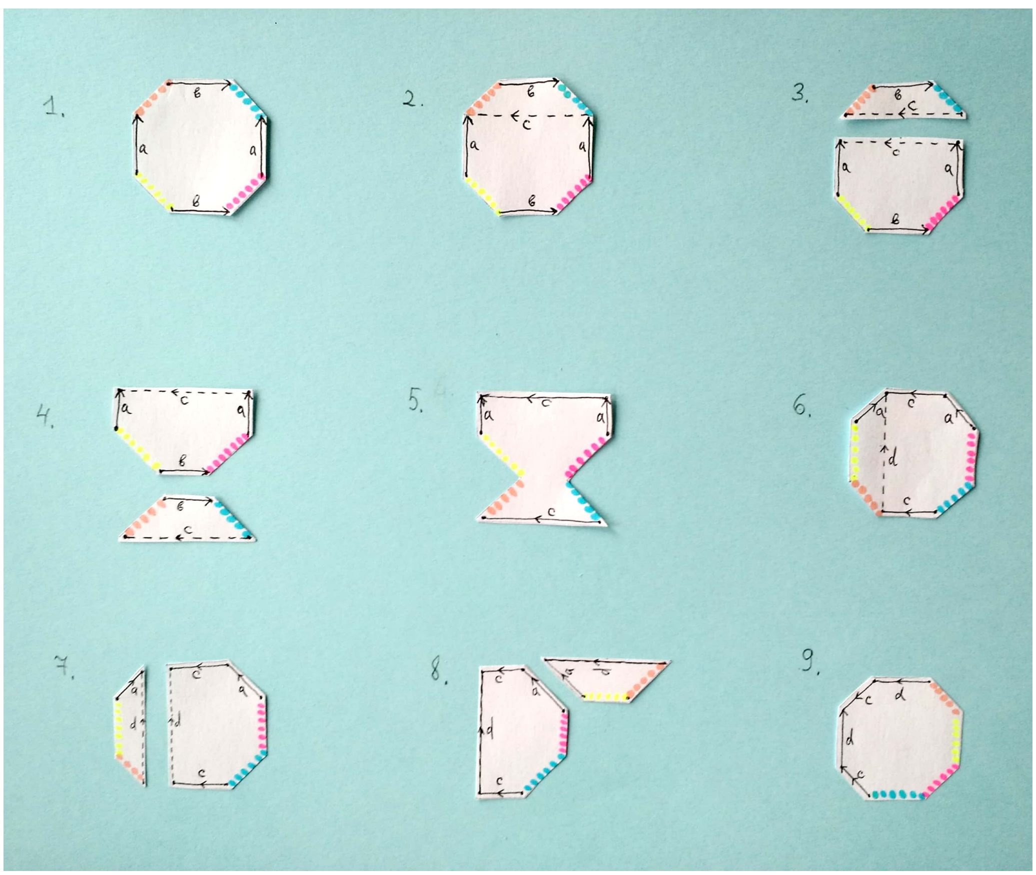



FIGURE 4. Manipulating a pair of opposing pairs to form $cdc^{-1}d^{-1}$.

**Exercise 7.** Show that $\mathbb{T}^2 \# \mathbb{P}^2$ is homeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.

There is another proof of the classification of surfaces (Conway's ZIP proof, or Zero Irrelevancy Proof). It argues inductively from a triangulation of the given surface, and the induction step is easier if one allows surfaces with boundary. If the triangulation consists of a single triangle, then the surface is a sphere with one hole. A case-by-case analysis shows that combining triangles to form a more complex surface will still yield a connected sum of tori and real projective planes, together with boundary components. Using the Exercise $\mathbb{T}^2 \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$, one can simplify the surface further so that it involves only tori or only real projective planes. For more on this proof, see "Symmetries of Things" (Chapter 8) by J. H. Conway, H. Burgiel, and C. Goodman-Strauss or the article "Conway's ZIP Proof" by G. Francis and J. Weeks.

4.5. **Euler characteristic.**

**Definition 4.11.** A quantity $\alpha$ is a *topological invariant* if $\alpha(X) = \alpha(Y)$ whenever $X$ and $Y$ are topologically equivalent.

**Definition 4.12.** A *cell* is a space whose interior is homeomorphic to the unit $n$-dimensional ball of the Euclidean space $\mathbb{R}^n$. The boundary of a cell must be composed of a finite number of lower-dimensional cells, the *faces of the cell*.

**Examples**

- 0-dimensional cell is a point
- 1-dimensional cell is an interval
- 2-dimensional cell is a polygon

**Definition 4.13.** A *CW complex* $K$ is a Hausdorff space $X$ together with a partition of $X$ into open cells (of perhaps varying dimension) that satisfies two additional properties:

(1) For each $n$-dimensional open cell $C$ in the partition of $X$, there exists a continuous map $f$ from the $n$-dimensional closed ball to $X$ such that
  - the restriction of $f$ to the interior of the closed ball is a homeomorphism onto the cell $C$, and
  - the image of the boundary of the closed ball is contained in the union of a finite number of elements of the partition, each having cell dimension less than $n$.

(2) A subset of $X$ is closed if and only if it meets the closure of each cell in a closed set.

We write $|K|$ to denote the topological space $X$ resulting from gluing cells together.

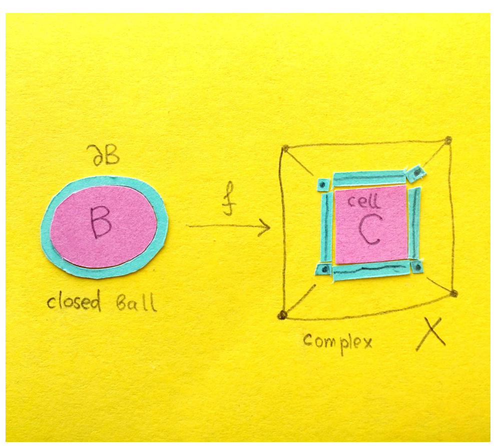



FIGURE 5. An example of a cell complex

**Definition 4.14.** The *Euler characteristic* of a finite complex $K$ is

$$\chi(K) = \sum_{j=0}^{\dim K} (-1)^j n_j$$

where $n_j$ is the number of $j$-cells in $K$.

**Theorem 4.15.** *Let $K$ and $L$ be 2-dimensional complexes. If $|K|$ and $|L|$ are compact connected surfaces and $|K|$ is homeomorphic to $|L|$ then $\chi(K) = \chi(L)$.*

**Definition 4.16.** Let $S$ be a compact connected surface and let $K$ be a complex such that $S$ is homeomorphic to $|K|$. Then we define $\chi(S) := \chi(K)$

**Exercise 8.** Let $S_1$ and $S_2$ be connected compact surfaces. Show that the Euler characteristic satisfies $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

**Exercise 9.** Compute $\chi(\mathbb{S}^2)$, $\chi(\mathbb{T}^2)$, and $\chi(\mathbb{P}^2\mathbb{R})$.

**Definition 4.17.** The *genus* of a compact connected orientable surface $X$ is defined as

$$g(X) := 1 - \frac{1}{2}\chi(X).$$

## 5. Riemann-Hurwitz formula

The following lemma will be proven in the next lecture.

**Lemma 5.1.** *Suppose that $\pi : M \to D$ is a covering of an open disk $D$. Then for each connected component $M_i$ of $M$ the restriction map $\pi : M_i \to D$ is a homeomorphism.*

**Theorem 5.2.** *(Riemann-Hurwitz formula) Let $f : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. Then*

$$\chi(X) = \deg(f)\,\chi(Y) - \sum_{p\in X}[\mathrm{mult}(f,p) - 1]$$

*Proof.* Let $B$ be the set of the bracnch points of map $f$. Then $f : X \setminus (f^{-1}(B)) \to Y \setminus B$ is a covering of degree $\deg(f)$. Let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a triangulation of $Y$. We choose the triangulation $\mathcal{T}$ so that the set $B$ is contained in the set of its vertices. Now will show that the triangulation $\mathcal{T}$ of $Y$ lifts to a triangulation $\mathcal{T}'$ of $X$ via the covering $f$. In other words there exists a triangulation $\mathcal{T}' = \{T_i'\}_{i=1}^{n'}$ of $X$ such that the image $f(T')$ of a triangle $T' \in \mathcal{T}'$ is a triangle in the triangulation $\mathcal{T}$.

Let $T$ be a triangle in the triangulation $\mathcal{T}$ and let $T^0$ be the set on interior points of $T$. There exists an open neighbourhood $T^\epsilon$ of $T^0$ such that: 1) $T^\epsilon$ does not contain vertices of $\mathcal{T}$, 2) edges of $T$ are contained in $T^\epsilon$, 3) $T^\epsilon$ is homeomorphic to an open disk. Note that the preimage $f^{-1}(T^\epsilon)$ does not contain ramification points of $f$. Therefore $f : f^{-1}(T^\epsilon) \to T^\epsilon$ is a covering. Let $f^{-1}(T^\epsilon) = \sqcup_i A_i'$ be the disjoint union of connected components. By Lemma 5.1 each restriction map $f : A_i' \to T^\epsilon$ is a homeomorphism. Therefore, we see that the preimage of $T$ consists of $\deg(f)$ triangles, these triangles have disjoint enterior points, disjoint (interior points of the) edges, and can might have common vertices. The union of all the preimages of triangles of $\mathcal{T}$ forms a desired triangulation $\mathcal{T}'$ of $X$.

Suppose that the triangulation $\mathcal{T}$ consists of $n$ triangles, $m$ edges, and $\ell$ vertices. Then $\mathcal{T}'$ consists of $\deg(f)\,n$ triangles, $\deg(f)\,m$ edges, and $\ell'$ vertices. Our next goal is to compute $\ell'$. We claim that

$$\ell' = \deg(f)\,\ell - \sum_{p\in X}(\mathrm{mult}(f,p) - 1). \tag{3}$$

Indeed, let $q \in Y$. Then

$$\sum_{p\in f^{-1}(q)} \mathrm{mult}(p) = \deg(f).$$

Or equivalently

$$\sum_{p\in f^{-1}(q)} (\mathrm{mult}(p) - 1) + \sharp(f^{-1}(q)) = \deg(f).$$

Here $\sharp(A)$ denotes the number of elements in a finite set $A$. Now we sum this identity over all vertices $q$ of the triagulation $\mathcal{T}$. Let $V$ be the set of all vertices in $\mathcal{T}$. Then

$$\sum_{q\in V}\left(\sum_{p\in f^{-1}(q)} (\mathrm{mult}(p) - 1) + \sharp(f^{-1}(q))\right) = \deg(f)\,\ell.$$

Note that the number of vertices in $\mathcal{T}'$ is $\ell' = \sum_{q\in V}\sharp(f^{-1}(q))$ and all ramification points of $f$ are contained in $f^{-1}(V)$. This proves the claim (3). Now as we have computed the numbers

$n', m'$, and $\ell'$ we find

$$n' - m' + \ell' = \deg(f)(n - m + \ell) - \sum_{p \in X} (\mathrm{mult}(p) - 1).$$

By Euler's formula we know that

$$n - m + \ell = \chi(Y) = 2 - 2g(Y) \quad \text{and} \quad n' - m' + \ell' = \chi(X) = 2 - 2g(X).$$

This finishes the proof of the theorem. □

**Exercise 10.** Let $X, Y$ be two compact connected Riemann surfaces. Suppose that $g(X) < g(Y)$. Show that any holomorphic map from $X$ to $Y$ is constant.

## 6. Normalization of algebraic curves

Let $P(X,Y) \in \mathbb{C}[X,Y]$ be a non-constant irreducible polynomial. We also assume that $P$ is not contained in $\mathbb{C}[X]$. Let $C_P$ be the corresponding affine curve consisting of smoth solutions of the equation $P(X,Y)=0$ as defined in Section 3.2. Let $\pi : C_P \to \mathbb{C}$ be the projection to the first coordinate.

**Theorem 6.1.** *There exists a compact Riemann surface $\widehat{C}_P$ such that $C_P \hookrightarrow \widehat{C}_P$ is an embedding and the set $\widehat{C}_P \setminus C_P$ is finite. Moreover, there exists a holomorphic map $\widehat{\pi} : \widehat{C}_P \to \mathbb{CP}^1$ such that the map $\pi$ is a restriction of $\widehat{\pi}$. Here we consider $\mathbb{CP}^1$ as the union $\mathbb{C} \cup \{\infty\}$.*

**Lemma 6.2.** *Let $U$ be a Riemann surface and $f : U \to D \setminus \{0\}$ be a holomorphic covering of a punctured disk. Then $U$ is biholomorphic to the disjoint union of punctured disks.*

# 7. Homotopy

## 7.1. Homotopy of Maps. Fundamental group.

**Definition 7.1.** Two continuous maps $f_1, f_2 : S \to M$ between manifolds $S$ and $M$ are *homotopic*, if there exists a continuous map

$$F : S \times [0,1] \to M$$

with

$$F|_{S\times\{0\}} = f_1 \text{ and } F|_{S\times\{1\}} = f_2.$$

We write $f_1 \approx f_2$.

**Definition 7.2.** Let $g_1, g_2 : [0,1] \to M$ be curves with

$$g_1(0) = g_2(0) = p_0,$$
$$g_1(1) = g_2(1) = p_1.$$

we say that $g_1$ and $g_2$ are *homotopic*, if there exists a continuous map

$$G : [0,1] \times [0,1] \to M$$

such that

$$G|_{\{0\}\times[0,1]} = p_0,\ G|_{\{1\}\times[0,1]} = p_1,$$
$$G|_{[0,1]\times\{0\}} = g_1,\ G|_{[0,1]\times\{1\}} = g_2.$$

We write again $g_1 \approx g_2$.

**Definition 7.3.** The *homotopy class* of a map $f$ is the equivalence class consisting of all maps homotopic to $f$. We denote it by $\{f\}$. Analogously, the homotopy class $\{g\}$ of a curve $g$ is the equivalence class consisting of all path with the same end-points, homotopic to $g$.

**Definition 7.4.** Let $g_1, g_2 : [0,1] \to M$ be curves with

$$g_1(1) = g_2(0)$$

(i.e. the terminal point of $g_1$ is the initial point of $g_2$). Then the product $g := g_2g_1$ is defined by

$$g(t) = \begin{cases} g_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Thus $g_1 \approx {g'}_1, g_2 \approx {g'}_2$ implies $g_1g_2 \approx {g'}_1{g'}_2$. Furthermore, we can define the multiplication of conjugacy classes by

$$\{g_1\} \cdot \{g_2\} = \{g_1g_2\}.$$

**Definition 7.5.** For any $p_0 \in M$, the *fundamental group* $\pi_1(M, p_0)$ is the group of homotopy classes of paths $g : [0,1] \to M$ with $g(0) = g(1) = p_0$, i.e. closed paths with $p_0$ as initial and terminal point.

**Theorem 7.6.** $\pi_1(M, p_0)$ *is a group with respect to the operation of multiplication of homotopy classes. The identity element is the class of the constant path* $g_0 \equiv p_0$.

**Lemma 7.7.** *For any* $p_0, p_1 \in M$*, the groups* $\pi_1(M, p_0)$ *and* $\pi_1(M, p_1)$ *are isomorphic.*

**Definition 7.8.** The abstract group $\pi_1(M)$ defined in view of Lemma 7.7 is called the *fundamental group* of $M$.

**Definition 7.9.** We say that $M$ is *simply-connected* if $\pi_1(M) = \{1\}$.

**Lemma 7.10.** *Let $f : M \to N$ be a continuous map, and $q_0 := f(p_0)$. Then $f$ induces a homomorphism*
$$f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$$
*of fundamental groups.*

**Exercise 11.** Determine the fundamental group of $S^1$.
Outline of the solution: Let $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta};$ with $\theta \in \mathbb{R}, 0 \leq \theta \leq 2\pi\}$. Then paths $\gamma_n$ in $\pi_1(S^1, 1)$ are given by
$$t \mapsto e^{2\pi int} \; t \in [0, 1]$$
for each $n \in \mathbb{Z}$.
Show that

1) $\gamma_n$ and $\gamma_m$ are not homotopic for $n \neq m$.
2) each $\gamma \in \pi_1(S^1, 1)$ is homotopic to some $\gamma_n$.

**Exercise 12.** Determine the fundamental group of the torus $S^1 \times S^1$.

**Theorem 7.11.** *Let $X$ be a connected compact orientable surface of genus $g$. Then the fundamental group of $X$ is isomorphic to the group with generators $a_1, b_1, \ldots, a_g, b_g$ and the only relation*
$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

*Proof.* The proof of this theorem is given in Section 1.2 of "Algebraic topology" by A. Hatcher. □

**Exercise 13.** Compute the fundamental group of a disc with $n$ punctures.

**Exercise 14.** Hang a picture on the wall using a round rope two nails in such a way that removing either of the two nails will make both the string and picture fall down. Can you hang a picture on three nails in such a way that removing either of the three nails will make the picture fall down.

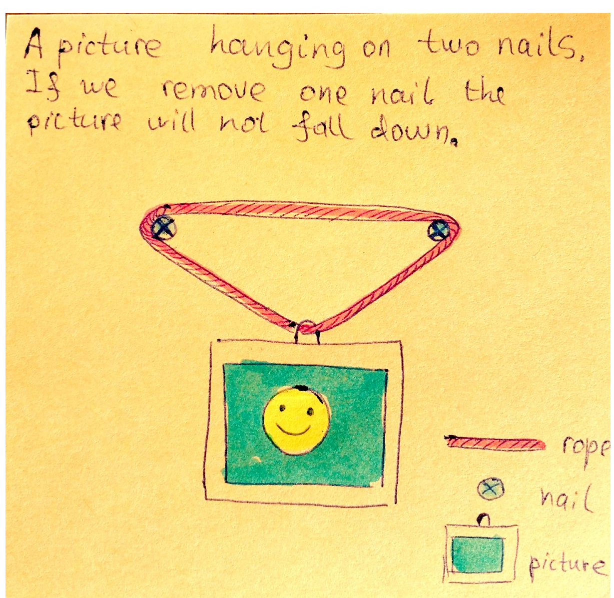



FIGURE 6. A pictire hanging on two nails.

7.2. **Coverings and homotopy groups.** In this subsection we assume $M$, $M'$ and $M''$ to be connected manifolds.

**Lemma 7.12.** *If $\pi : M' \to M$ is a covering, then each point of $M$ is covered the same number of times, i.e. $\pi^{-1}(x)$ has the same number of elements for each $x \in M$.*

**Theorem 7.13.** *Let $\pi : M' \to M$ be a covering, $S$ a simply-connected manifold, and $f : S \to M$ a continuous map. Then there exists a continuous $f' : S \to M'$ with*

$$\pi \circ f' = f.$$

**Definition 7.14.** An $f'$ as in Theorem 7.13 is called a *lift* of $f$.

**Lemma 7.15.** *Let $\pi : M' \to M$ be a covering, $p_0 \in M$, ${p'}_0 \in \pi^{-1}(p_0)$, and $g : [0,1] \to M$ a curve with $g(0) = p_0$. Then $g$ can be lifted (as in Definition 7.14) to a curve $g' : [0,1] \to {p'}_0$, so that*

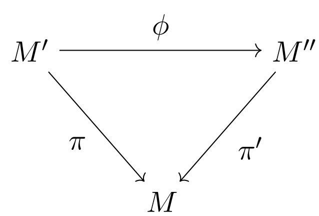
$$\pi \circ g' = g.$$

*Further, $g'$ is uniquely determined by the choice of its initial point ${p'}_0$.*

**Corollary 7.16.** *Let $\pi : M' \to M$ be a covering, $g : [0,1] \to M$ a curve with $g(0) = g(1) = p_0$, and $g' : [0,1] \to M'$ a lift of $g$. Suppose $g$ is homotopic to the constant curve $\gamma(t) \equiv p_0$. Then $g'$ is closed and homotopic to the constant curve.*

**Lemma 7.17.** *$G_\pi := \{\{g\} : g'$ is closed$\}$ is a subgroup of $\pi_1(M, p_0)$.*

**Definition 7.18.** Let $\pi : M' \to M$ and $\pi' : M'' \to M$ be two coverings of $M$. We say that $\pi$ and $\pi'$ are equivalent if there exists a homeomorphism $\phi : M' \to M''$ such that $\pi = \pi' \circ \phi$. Equivalently, the following diagram commutes:




**Theorem 7.19.** ***(Galois theory for coverings)*** *The group $\pi_1(M')$ is isomorphic to $G_\pi$, and we obtain in this way a bijective correspondence between conjugacy classes of subgroups of $\pi_1(M)$ and equivalence classes of coverings $\pi : M' \to M$.*

**Corollary 7.20.** *If $M$ is simply-connected, then every covering $M' \to M$ is a homeomorphism.*

**Corollary 7.21.** *Every connected manifold $M$ has a covering $\widetilde{\pi} : \widetilde{M} \to M$ by a simply connected manifold $\widetilde{M}$.*

**Definition 7.22.** The covering $\widetilde{M}$ of $M$ with $\pi_1(\widetilde{M}) = 1$ - which exists by Corollary 7.20 - is called the *universal covering* of $M$.

**Lemma 7.23.** *Let $X$ be a complex manifold and let $\pi : M \to X$ be a covering. Then $M$ has a natural structure of a complex manifold.*

**Theorem 7.24.** ***(Uniformization Theorem)*** *Any simply-connected Riemann surface is isomorphic either to $\mathbb{CP}^1$, or to $\mathbb{C}$, or to $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.*

**Definition 7.25.** Let $\pi : M' \to M$ be a local homeomorphism. Then a homeomorphism $\phi : M' \to M'$ is called a *covering transformation* ( or a *deck transformation*) if $\pi \circ \phi = \pi$. The covering transformations form a group.

*Example.* Let $\Lambda \subset \mathbb{C}$ be a lattice. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the natural projection. Then for any $\lambda \in \Lambda$ the map $z \mapsto z + \lambda$ is a covering transformation of $\pi$.

**Exercise 15.** Construct a manifold $M$ with a (nontrivial) covering map $\pi : S^3 \to M$.
*Hint:* The group SO(4) acts on $S^3$ considered as the unit sphere in $\mathbb{R}^4$. Find a discrete subgroup $\Gamma$ of SO(4) for which no $\gamma \in \Gamma \setminus \{\text{identity}\}$ has a fixed point in $S^3$.

**Exercise 16.** Consider the group

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 3 \right\}.$$

This group acts on

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

via

$$z \mapsto \frac{az + b}{cz + d}.$$

1) Show that if $\gamma \in \Gamma$ is different from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\gamma$ has no fixed points in $\mathfrak{H}$.
1) Interpret $\Gamma$ as the group of covering transformations associated with a manifold $\mathfrak{H}/\Gamma$ and a covering $\pi : \mathfrak{H} \to \mathfrak{H}/\Gamma$. Construct different coverings of $\mathfrak{H}/F$ associated with conjugacy classes of subgroups of $\Gamma$.

**Definition 7.26.** Let $\pi : (X, x_0) \to (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set

$$G := \pi_1(Y, y_0), \qquad H := \pi_*\big(\pi_1(X, x_0)\big) \le G,$$

and write $\mathrm{Deck}(\pi)$ for the group of deck (covering) transformations. Let $F = \pi^{-1}(y_0)$ be the fiber over $y_0$. Given a loop $\gamma : [0, 1] \to Y$ based at $y_0$, and a point $x \in F$, there is a unique lift $\widetilde{\gamma}_x$ of $\gamma$ starting at $x$, i.e. $\widetilde{\gamma}_x(0) = x$ and $\pi \circ \widetilde{\gamma}_x = \gamma$. The endpoint of the lift $\widetilde{\gamma}_x$ lands again in the fiber $F$. Therefore loops in $Y$ induce permutations in the fiber $F$. Taking homotopy classes, this defines the *monodromy representation*

$$\rho : G \to \mathrm{Sym}(F), \quad \rho([\gamma])(x) := \widetilde{\gamma}_x(1) \in F.$$

**Theorem 7.27.** *The monodromy representation has the following properties.*

(1) *The image of monodromy representation $\rho(G)$ acts transitively on the fiber $F$.*
(2) $H = \mathrm{Stab}_G(x_0) = \{g \in G : \rho(g)(x_0) = x_0\}$.
(3) *The coset space $G/H$ is in bijection with $F$.*
(4) *The kernel of the monodromy representation can be described as*

$$\ker(\rho) = \bigcap_{x \in F} \mathrm{Stab}_G(x) = \bigcap_{g \in G} gHg^{-1}.$$

(5) *If a deck transformation $\tau \in \mathrm{Deck}(\pi)$ fixes some $x \in X$, then $\tau = \mathrm{id}_X$. Show that every $\tau \in \mathrm{Deck}(\pi)$ induces a bijection $\tau|_F : F \to F$ and conclude that*

$$\Phi : \mathrm{Deck}(\pi) \to \mathrm{Sym}(F), \qquad \Phi(\tau) = \tau|_F,$$

*is injective.*
(6) $\Phi\big(\mathrm{Deck}(\pi)\big)$ *is the centralizer of monodromy, that is,*

$$\begin{aligned}\Phi\big(\mathrm{Deck}(\pi)\big) &= \mathrm{Cent}_{\mathrm{Sym}(F)}\big(\rho(G)\big) \\ &= \{\sigma \in \mathrm{Sym}(F) : \sigma\rho(g) = \rho(g)\sigma \text{ for all } g \in G\}.\end{aligned}$$

**Exercise 17.** Let $F$ be a set and let $A \leq \mathrm{Sym}(F)$ act transitively on $F$. Let $C = \mathrm{Cent}_{\mathrm{Sym}(F)}\big(A\big)$ be the centralizer of $A$ in $\mathrm{Sym}(F)$. Show that

(1) $C$ acts freely on $F$.
(2) $C$ acts transitive on $F$ if and only if $A$ acts freely on $F$.

**Theorem 7.28.** *Let $\pi : (X, x_0) \to (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set*

$$G := \pi_1(Y, y_0), \qquad H := \pi_*\big(\pi_1(X, x_0)\big) \leq G,$$

*and write* $\mathrm{Deck}(\pi)$ *for the group of deck (covering) transformations. Let $d = |\pi^{-1}(y_0)| = \deg(\pi)$ and let $\rho : G \to \mathrm{Sym}(\pi^{-1}(y_0)) \cong S_d$ be the monodromy representation on the fiber $F = \pi^{-1}(y_0)$. The following are equivalent (any may be taken as the definition of a Galois (regular/normal) covering):*

(1) $\mathrm{Deck}(\pi)$ *acts transitively on the fiber $F$.*
(2) *$H$ is a normal subgroup of $G$.*
(3) *There exists a free, proper action of a discrete group $\Gamma$ on $X$ with $Y \cong X/\Gamma$ and $\pi$ the quotient map.*
(4) *The monodromy image $\rho(G) \subset S_d$ acts freely and transitively on the fiber $F$.*

*When these equivalent conditions hold, we have group isomorphisms*

$$\mathrm{Deck}(\pi) \cong \Gamma \cong G/H \cong \rho(G),$$

*and* $|\mathrm{Deck}(\pi)| = \deg(\pi)$.

**Definition 7.29.** The covering that satisfies the equivalent conditions of Theorem 7.28 is called the Galois (regular/normal) covering.

**References:**

1. A. Hatcher , Algebraic topology, Cambridge University Press, 2002.
2. J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.

## 8. Compact Riemann surfaces and complex algebraic curves

**Theorem 8.1.** *Let $X$ be a connected compact Riemann surface. Let $P_1, \ldots, P_n$ be a collection of pairwise distinct points on $X$. Then there exists a holomorphic function $g : X \to \mathbb{CP}^1$ such that*

$$g(P_i) = \begin{cases} \infty, i = 1; \\ \neq \infty, i = 2, \ldots, n. \end{cases}$$

**Theorem 8.2.**

**Theorem 8.3.**

## 9. Homology and cohomology

### 9.1. Singular homology.

**Definition 9.1.** The *standard $n$-simplex* is the set

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0\}.$$

**Definition 9.2.** A *singular $n$-simplex* is a continuous map $\sigma_n$ from the standard $n$-simplex $\Delta^n$ to a topological space $X$.

**Definition 9.3.** Let $v_0, \dots, v_n \in \mathbb{R}^n$ be $n+1$ points in general position. We denote by $[v_0, \dots, v_n]$ the simplex

$$[v_0, \dots, v_n] = \{t_0 v_0 + \ldots + t_n v_n \mid \sum_i t_i = 1 \text{ and } t_i \geq 0\}.$$

**Definition 9.4.** A *canonical homeomorphism* from the standard simplex $\Delta^n$ to any other $n$-simplex $[v_0, \dots, v_n]$ is defined as

$$(t_0, \dots, t_n) \mapsto \sum_i t_i \, v_i.$$

*Remark.* The vertices in the standard simplex and in the simplex $[v_0, \dots, v_n]$ are numbered.

**Definition 9.5.** A *singular $n$-chain* is a formal sum of singular simplices $\sum_i \sigma_i$. The group of singular $n$-chains on $X$ is denoted $C_n(X)$.

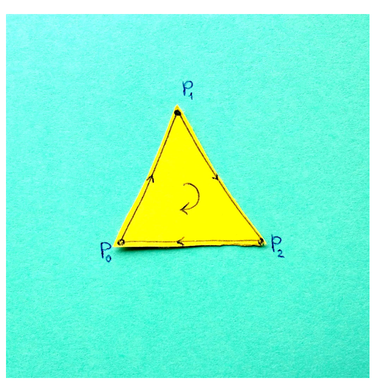
**Definition 9.6.** A *boundary map* is defined by

$$\partial_n[p_0, \dots, p_n] = \sum_{k=0}^{n} (-1)^k [p_0, \dots, p_{k-1}, p_{k+1}, \dots, p_n], \qquad n \geq 1$$
$$\partial_0[p_0] = 0.$$

**Example.**

$$\partial_2[p_0, p_1, p_2] = [p_1, p_2] - [p_0, p_2] + [p_0, p_1].$$

Figure 7. Boundary of a simplex $[p_0, p_1, p_2]$.



**Definition 9.7.** The *group of singular $n$-cycles* is defined

$$Z_n(X) := \ker(\partial_n).$$

The *group of singular $n$-boundaries* is defined

$$B_n(X) := \mathrm{Im}(\partial_{n+1}).$$

**Proposition 9.8.** *We have*

$$\partial_n \circ \partial_{n+1} = 0,$$

*or equivalently, $B_n \subseteq Z_n$.*

**Definition 9.9.** The $n$-th *singular homology group* of $X$ is

$$H_n(X) := Z_n(X)/B_n(X).$$

**Proposition 9.10.** *If $X$ is non-empty and path connected then $H_0(X) \cong \mathbb{Z}$.*

**Definition 9.11.** For a map $f : X \to Y$, an *induced homomorphism* $f_\# : C_n(X) \to C_n(Y)$ is defined by composing each singular $n$-simplex $\sigma : \Delta^n \to X$ with $f$ to $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$.

**Proposition 9.12.** *The chain map $f_\#$ between the groups of singular $n$-chains $C_n(X)$ and $C_n(Y)$ induces homomorphism $f_* : H_n(X) \to H_n(Y)$ between homology groups.*

**Theorem 9.13.** *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

*Proof.* The proof can be found in "Introduction to algebraic topology" by A. Hatcher. □

**Theorem 9.14.** *Let $X$ be a compact, connected, orientable surface of genus $g$. Then*

$$\begin{aligned} &H_0(X) \cong \mathbb{Z} \\ &H_1(X) \cong \mathbb{Z}^{2g} \\ &H_2(X) \cong \mathbb{Z} \\ &H_n(X) \cong \{0\} \textit{ for } n > 2. \end{aligned}$$

**Exercise 18.** Construct an example of a compact surface $X$ and a closed path $\gamma$ on it such that $\gamma$ is homologous to zero but not homotopic to a point.

9.2. **Intersection pairing.** Let $X$ be a smooth oriented surface.

**Definition 9.15.** Let $\gamma_1$ and $\gamma_2$ be two smooth curves on $X$ intersecting transversally at the point $P$. The *intersection number* of $\gamma_1$ and $\gamma_2$ at the point $P$ is

$$(\gamma_1 \circ \gamma_2)_P := \text{sign of the determinant of a basis } \gamma_1'(P), \gamma_2'(P) \text{ on the tangent space } T_P(X).$$

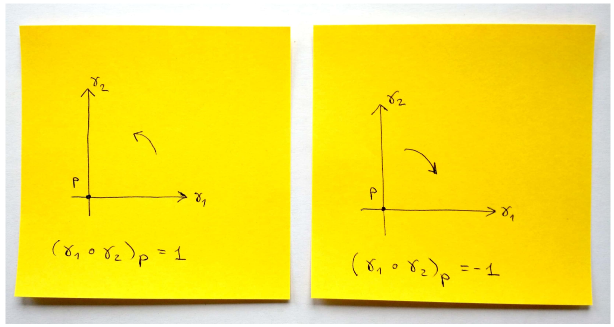



FIGURE 8. Intersection number.

**Definition 9.16.** Let $\gamma_1, \gamma_2$ be two smooth cycles intersecting transversally at the finite set of points. The *intersection number* of $\gamma_1$ and $\gamma_2$ is defined by

$$\gamma_1 \circ \gamma_2 = \sum_{P \in (\gamma_1 \cap \gamma_2)} (\gamma_1 \circ \gamma_2)_P.$$

**Proposition 9.17.** *The intersection number of any boundary $\beta$ with any cycle $\gamma$ vanishes.*

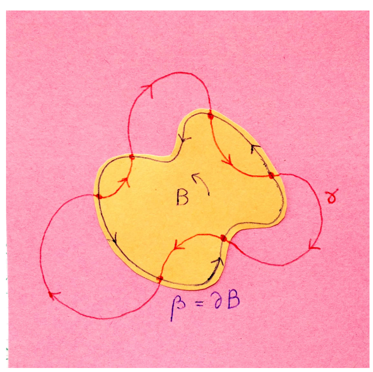



FIGURE 9. Intersection number of a boundary $\beta$ with a cycle $\gamma$.

**Theorem 9.18.** *The intersection number is a bilinear skew-symmetric form $\circ : H_1(X) \times H_1(X) \to \mathbb{Z}$.*

**Exercise 19.** Let $X$ be Riemann surface of genus $g$ and let $\Pi_g$ be a planar model of $X$ with the symbol $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Show that the intersection numbers of the cycles $a_1, b_1, \dots a_g, b_g$ are as follows:

$$a_i \circ b_j = \delta_{i,j} \qquad a_i \circ a_j = b_i \circ b_j = 0. \tag{4}$$

**Definition 9.19.** A homology basis $a_1, \dots a_g, b_1, \dots b_g$ of a compact connected Riemann surface is called a *canonical basis of cycles* if (4) holds.

Consider a $2g \times 2g$ integral matrix

$$J_g := \begin{pmatrix} 0_g & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0_g \end{pmatrix}.$$

Consider the group of matrices

$$\mathrm{Sp}_{2g}(\mathbb{Z}) := \{M \in \mathrm{M}_{2g\times 2g}(\mathbb{Z}) \mid MJM^t = J\}.$$

This group is called the *symplectic group* of genus $g$.

**Exercise 20.** Let $X$ be a Riemann surface and let $a_1, \dots, a_g, b_1, \dots, b_g$ be a canonical basis of $H_1(X)$. Show that $a'_1, \dots, a'_g, b'_1, \dots b'_g$ is another canonical basis if and only if

$$\begin{bmatrix} a'_1 \\ \vdots \\ a'_g \\ b'_1 \\ \vdots \\ b'_g \end{bmatrix} = M \begin{bmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{bmatrix}.$$

for some $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$.

## 10. Calculus on Riemann surfaces

10.1. **Differential forms.** Let $M$ be a differentiable manifold of dimension $d$. We denote by $TM$ the tangent bundle of $M$.

**Definition 10.1.** For a vector space $V$ we denote by $\Lambda^k V$ the space of $k$-linear alternating maps $V^k \to \mathbb{R}$.

**Definition 10.2.** A *(differential) $k$-form* on a manifold $M$ is a (smooth) section of the bundle $\Lambda^k TM$. The space of differential $k$-forms on $M$ is denoted by $\Omega^k(M)$.

In local coordinates $x^1, \dots, x^d$ a differential $k$-form $\omega$ is an object of the form

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Here $\omega_{i_1 \cdots i_k}$ are real-valued differentiable functions.

**Definition 10.3.** The *exterior derivative* of $\omega$ is the form of degree $k+1$

$$\mathrm{d}\omega := \sum_{i_0=1}^{d} \sum_{i_1 < \dots < i_k} \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^{i_0}} \mathrm{d}x^{i_0} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

**Exercise 21.** Show that $\mathrm{d}^2 = 0$.

**Definition 10.4.** A differential form $\omega$ is said to be *closed* if $\mathrm{d}\omega = 0$. A differential form $\omega$ is said to be *exact* if $\omega = \mathrm{d}\alpha$ for some differential form $\alpha$.

**Definition 10.5.** The $k$-th de Rham cohomology group of $M$ is defined as

$$H^k_{\mathrm{dR}}(M, \mathbb{R}) := \frac{\text{closed forms of degree } k}{\text{exact forms of degree } k}.$$

**Exercise 22.** Compute $H^i_{\mathrm{dR}}$ of $S^1$.

10.2. **Integration of differential forms. De Rham cohomology.** Let $M$ be a differentiable manifold of dimension $d$. Let $U \subset M$ be an open chart with local coordinates $x^1, \dots x^d$. Let $\omega$ be a differential form of degree $k$ and let $S \subset U$ be a submanifold of dimension $k$. Suppose that $S$ is parametrized by an open set $V \subset \mathbb{R}^k$

$$S(v) = (x^1(v), \dots, x^d(v)), \quad v \in V \subset \mathbb{R}^k.$$

We define

$$\int_S \omega := \int_V \sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k}(S(v)) \frac{\partial(x^{i_1}(v), \dots, x^{i_k}(v))}{\partial(v^1, \dots, v^k)} \mathrm{d}v^1 \cdots \mathrm{d}v^k.$$

This definition is extended by linearity to $k$-chains.

**Theorem 10.6.** *(Stokes) Let $D$ be a $k$-chain and $\omega$ be a $k-1$ form. Then*

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

**Corollary 10.7.** *There exists a pairing $H^k_{\mathrm{dR}}(M) \times H_k(M) \to \mathbb{R}$*

$$(\omega, D) \mapsto \int_D \omega$$

**Theorem 10.8.** *(de Rham)*

$$H^k_{\mathrm{dR}} \cong \mathrm{Hom}(H_k(M) \otimes \mathbb{R}, \mathbb{R}).$$

This is a deep theorem in differential geometry and its proof goes beyond the scope of our course. An interested reader can find a proof of this result in "Principles of Algebraic Geometry" by J. Harris and P. Griffiths.

**Proposition 10.9.** *Let $S$ be a compact, connected, oriented, smooth surface. Then the map*

$$\alpha \mapsto \int_S \alpha$$

*is an isomorphism from $H^2_{\mathrm{dR}}(S)$ to $\mathbb{R}$.*

**Proposition 10.10.** *Let $S$ be a compact, connected, oriented, smooth surface. Let $\gamma$ be a closed 1-cycle on $S$. Then there exists a $C^\infty$ 1-form $\eta_\gamma$ such that*

(1) $d\eta_\gamma = 0$
(2) $\int_\gamma \beta = \int_S \beta \wedge \eta_\gamma$ *for any closed* 1*-form* $\beta$.
(3) $\int_\alpha \eta_\gamma = \alpha \circ \gamma$ *for any closed cycle* $\alpha$.

**References:**

A. Hatcher , Algebraic topology, Cambridge University Press, 2002.
J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.
J. Harris and P. Griffiths, Principles of Algebraic Geometry, 1978.

## 11. Harmonic and holomorphic differential forms on Riemann surfaces

11.1. **Decomposition of 1-forms.** Let $X$ be a Riemann surface and let $\alpha$ be a complex $C^\infty$ 1-form defined on an open subset $U$ of $X$. Let $(V, z), V \subset U$ be a holomorphic chart. Then $(x, y) := (\Re(z), \Im(z))$ is a real local coordinate. We have

$$\begin{aligned}\alpha =& f\,\mathrm{d}x + g\,\mathrm{d}y = \\ & \frac{1}{2}(f - ig)(\mathrm{d}x + i\mathrm{d}y) + \frac{1}{2}(f + ig)(\mathrm{d}x - i\mathrm{d}y) = \\ & \frac{1}{2}(f - ig)\,\mathrm{d}z + \frac{1}{2}(f + ig)\,\mathrm{d}\overline{z}\end{aligned}$$

**Definition 11.1.** For a 1-form $\alpha = \phi\mathrm{d}z + \psi\mathrm{d}\overline{z}$ we define

$$\begin{aligned}\alpha^{(1,0)} :=& \phi\mathrm{d}z \\ \alpha^{(0,1)} :=& \psi\mathrm{d}\overline{z}.\end{aligned}$$

**Lemma 11.2.** *The forms $\alpha^{(1,0)}$ and $\alpha^{(0,1)}$ do not depend on the choice of the local holomorphic coordinate $z$ and therefore they are $C^\infty$ forms defined globally on $U$.*

**Definition 11.3.** Let $z$ be a holomorphic coordinate on $V$ and $f : V \to \mathbb{C}$ be a $C^\infty$ function. Let $x = \Re(z)$ and $y = \Im(z)$. We define

$$\begin{aligned}\frac{\partial}{\partial z} f :=& \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f \\ \frac{\partial}{\partial \overline{z}} f :=& \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f\end{aligned}$$

**Lemma 11.4.** *The operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} : C^\infty(V) \to C^\infty(V)$ do not depend on the choice of the local holomorphic coordinate $z$ and therefore they are defined globally on $C^\infty(X)$.*

11.2. **Harmonic and holomorphic 1-forms.**

**Definition 11.5.** A differential 1-form $\omega$ on a Riemann surface $X$ is called *holomorphic* if in any local chart it is represented as

$$\omega = h(z)\,\mathrm{d}z$$

where $h(z)$ is holomorphic. The differential 1-form $\overline{\omega} = \overline{h(z)}\,\mathrm{d}\overline{z}$ is called *anti-holomorphic.*

**Definition 11.6.** The *conjugation operator* $* : \Omega^1(X) \to \Omega^1(X)$ is defined as

$$\begin{aligned}\omega \mapsto& -i\omega^{(1,0)} + i\omega^{(0,1)} \\ f\mathrm{d}z + g\mathrm{d}\overline{z} \mapsto& -if\mathrm{d}z + ig\mathrm{d}\overline{z}.\end{aligned}$$

**Exercise 23.**

$$** = -1.$$

**Definition 11.7.** Consider the operator

$$\begin{aligned}\Delta :& \Omega^0 \to \Omega^2 \\ & f \to -\mathrm{d} * \mathrm{d}f.\end{aligned}$$

**Definition 11.8.** A 1-form $\alpha$ is called *co-closed* if $d^*\alpha = 0$. A 1-form $\alpha$ is called *co-exact* if $\alpha = d * \eta$ for some 1-form $\eta$.

**Definition 11.9.** A function $f$ on a Riemann surface is called harmonic if $\Delta f = 0$.

**Definition 11.10.** A 1-form $\alpha$ is said to be *harmonic* if it is locally of the form $\alpha = \mathrm{d}f$ with $f$ a harmonic function.

**Exercise 24.** A 1-form $\eta$ is harmonic if and only if $\mathrm{d}\eta = \mathrm{d}^*\eta = 0$.

**Definition 11.11.** A 1-form $\alpha$ is said to be *holomorphic* if it is locally of the form $\alpha = \mathrm{d}f$ with $f$ a holomorphic function.

**Exercise 25.** a) A 1-form $\eta$ is harmonic if and only if it is of the form $\eta = \alpha_1 + \overline{\alpha}_2$ with $\alpha_1$ and $\alpha_2$ holomorphic.
b) A 1-form $\alpha$ is holomorphic if and only if it is of the form $\alpha = \eta + i * \eta$ with $\eta$ harmonic.

**Theorem 11.12.** *("The main theorem for compact Riemann surfaces") Let $X$ be a compact connected Riemann surface and let $\rho$ be a 2-form on $X$. There is a solution $f \in \Omega^0(X)$ to the equation*

$$\Delta f = \rho$$

*if and only if*

$$\int_X \rho = 0.$$

*The solution $f$ is unique up to an additive constant.*

**Theorem 11.13.** *Let $\omega$ be a 1-form on a compact connected Riemann surface $X$. Then there exist $f, g \in \Omega^0(X)$ end a harmonic 1-form $\eta$ such that*

$$\omega = \mathrm{d}f + *\mathrm{d}g + \eta.$$

**Theorem 11.14.** *For every closed 1-form $\alpha$ there exists a unique harmonic form $\alpha'$ such that*

$$\int_\gamma \alpha = \int_\gamma \alpha'$$

*for all closed cycles $\gamma$.*

**Theorem 11.15.** *Let $X$ be a compact Riemann surface of genus $g$. Then $H^1_{\mathrm{dR}}$ is isomorphic to the space of harmonic 1-forms on X. Moreover, it is a $\mathbb{C}$-vector space of dimension $2g$.*

**References:**

J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.

## 12. Periods of holomorphic differentials. Jacobian

Recall, that a differential 1-form $\omega$ on a Riemann surface $X$ is holomorphic if and only if in any local chart it is represented as

$$\omega = h(z)dz$$

where $h(z)$ is holomorphic function. The 1-form $\overline{\omega}$ is a called *anti-holomorphic*. Holomorphic differentials form a complex vector space. We will denote this space by $\Omega^1_{\mathrm{hol}}(X)$.

**Theorem 12.1.** *Let $X$ be a compact Riemann surface of genus $g$. Then*

$$\dim \Omega^1_{\mathrm{hol}}(X) = g.$$

**Examples**

- Let $X = \mathbb{C}/\Lambda$ be a torus. Then $\omega = dz$ is a holomorphic differential on $X$.
- Let $g \geq 0$ be an integer. Set $N = 2g + 2$ or $N = 2g + 1$. Let $C$ be a hyperelliptic affine curve given by the equation

$$y^2 = \prod_{i=1}^{N}(x - x_i),$$

where $x_i, i = 1, \dots, n$ are distinct complex number. The curve $C$ is isomorphic to a compact Riemann surface of genus $g$ with 1 or 2 punctures, depending on the parity of $N$. We denote this compact Riemann surface by $X$.

**Lemma 12.2.** *The differentials*

$$\omega_j = \frac{x^{j-1}\,dx}{y} \quad j = 1, \dots, g$$

*form a basis of the space of holomorphic differentials on $X$.*

**Theorem 12.3.** *(Riemann bilinear identity) Let $X$ be a compact Riemann surface of genus $g$ with a canonical homology basis $\{a_i, b_i\}_{i=1}^g$. Let $\omega, \omega'$ be two closed differentials on $X$ and let*

$$A_i := \int_{a_i} \omega \qquad B_i := \int_{b_i} \omega$$

$$A'_i := \int_{a_i} \omega' \qquad B'_i := \int_{b_i} \omega'$$

*be their periods. Then*

$$\int_X \omega \wedge \omega' = \sum_{i=1}^{g}(A_i B'_i - A'_i B_i)$$

**Lemma 12.4.** *Let $\omega$ be a non-zero holomorphic differential on $X$. Then periods $\{A_j, B_j\}_{j=0}^g$ of $\omega$ with respect to a canonical homology basis $\{a_i, b_i\}_{i=1}^g$ satisfy*

$$\mathrm{Im}(\sum_{j=0}^{g} A_j\, \overline{B}_j) < 0.$$

**Corollary 12.5.** *If all periods of a holomorphic differential $\omega$ are real, then $\omega \equiv 0$.*

**Definition 12.6.** Let $\{a_i, b_i\}_{i=1}^g$ be a canonical basis of $H_1(X, \mathbb{Z})$. The unique basis $\{\omega_k\}_{k=1}^g$ of $\Omega^1_{\mathrm{hol}}(X)$ that satisfies

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}$$

is called the *dual canonical basis.*

**Definition 12.7.** Let $\{a_i, b_i\}_{i=1}^g$ be a canonical homology basis of a compact Riemann surface $X$ and $\{\omega_k\}_{k=1}^g$ be the dual canonical basis of $\Omega^1_{\text{hol}}(X)$. The matrix

$$\mathcal{P} = \{B_{jk}\}_{j,k=1}^g \quad B_{jk} = \frac{1}{2\pi i} \int_{b_j} \omega_k$$

is called the *period matrix* of $X$.

**Theorem 12.8.** *The period matrix $\mathcal{P}$ is symmetric and its imaginary part is positive definite.*

**Exercise 26.** Let $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ be period matrices of the Riemann surface $X$ associated to homology basises $(a, b)$ and $(\widetilde{a}, \widetilde{b})$. Furthermore, let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ such that $\begin{pmatrix} \widetilde{a} \\ \widetilde{b} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ (see Exercise 20). Show that $\widetilde{\mathcal{P}} = (D\mathcal{P} + C)(B\mathcal{P} + A)^{-1}$.

**Definition 12.9.** The lattice

$$\Lambda := \{2\pi i \vec{m} + 2\pi i \mathcal{P} \vec{n} \mid \vec{m}, \vec{n} \in \mathbb{Z}^g\}$$

is called the *period lattice* of $X$.

**Definition 12.10.** The complex torus

$$\mathrm{Jac}(X) := \mathbb{C}^g / \Lambda$$

is called the *Jacobian* of $X$.

**Definition 12.11.** The *Abel map* $\mathcal{A} : X \to \mathrm{Jac}(X)$ is defined by

$$P \mapsto \mathcal{A}(P) = \int_{P_0}^{P} \vec{\omega}$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_g)^t$ is the vector composed by the dual canonical basis and $P_0 \in X$ is an arbitrarily chosen base point.

**References:**

J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.

# 13. Divisor group. Abel map and Abel theorem

## 13.1. **Meromorphic differentials.**

**Definition 13.1.** A differential $\Omega$ is called *meromorphic* if in any local chart $z : U \to \mathbb{C}$ it is of the form $\Omega = g(z)dz$, where $g$ is a meromorphic function.

Let $z$ be a local parameter at the point $P$, $z(P) = 0$ and

$$\Omega = \sum_{k=N(P)}^{\infty} g_k \, z^k \, dz \quad \text{for some } N(P) \in \mathbb{Z},$$

the representation of the differential $\Omega$ at $P$. The numbers $N(P)$ and $g_{-1}$ do not depend on the choise of the local parameter and are characteristics of $\Omega$ only.

**Definition 13.2.** The number $N(P)$ is called the *order of* $\Omega$ *at the point* $P$. The coefficient $g_{-1}$ is called the *residue* of $\Omega$ at $P$.

We have

$$\mathrm{res}_P \Omega = g_{-1} = \frac{1}{2\pi i} \int_\gamma \Omega$$

where $\gamma$ is a small closed loop going around $P$ in the positive direction (counterclockwise).

**Lemma 13.3.** *Let* $\Omega$ *be a meromorphic differential on a compact Riemann surface* $X$*. Then*

$$\sum_{P \in X} \mathrm{res}_P(\Omega) = 0.$$

## 13.2. **Divisors.**

**Definition 13.4.** A formal linear combination

$$D = \sum_{j=1}^{N} n_j P_j, \quad n_j \in \mathbb{Z}, \; P_j \in X$$

is called *a divisor* on the Riemann surface $X$. The sum

$$\deg D = \sum_{j=1}^{N} n_j$$

is called the *degree* of $D$. The set of all divisors forms the abelian group Div(X).

**Definition 13.5.** Let $f$ be a meromorphic function on $X$. Suppose that the zeroes of $f$ are $P_1, \ldots, P_M$ with multiplicities $m_1, \ldots, m_M$ and the poles of $f$ are $Q_1, \ldots, Q_N$ with multiplicities $n_1, \ldots, n_N$. The divisor

$$D = \sum m_i P_i - \sum n_i Q_i =: (f)$$

is called *the divisor of* $f$.

We denote

$$\mathrm{ord}_P(f) = \begin{cases} m & \text{if } P \text{ is a zero of order } m \\ -m & \text{if } P \text{ is a pole of order } m \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(f) = \sum_{P \in X} \mathrm{ord}_P(f) \, P.$$

**Definition 13.6.** A divisor $D$ is called *principal* if there exists a meromorphic function $f$ with $(f) = D$.

**Definition 13.7.** Two divisors $D$ and $D'$ are called *linearly equivalent* if $D - D'$ is principal.

**Definition 13.8.** The divisor of a meromorphic differential $\Omega$ is

$$(\Omega) = \sum_{P \in X} N(P)\, P$$

where $N(P)$ is the order of $\Omega$ at the point $P$.

Since the quotient of two meromorphic differentials $\Omega_1/\Omega_2$ is a meromorphic function, any two divisors of meromorphic differentials are linearly dependent.

**Definition 13.9.** The corresponding divisor class is called *canonical.* We will denote it by $C$.

**Definition 13.10.** A divisor $D = \sum n_j P_j$ is called *positive (or effective)* if all $n_j \geq 0$.

**Exercise 27.** Let $f$ be a meromorphic function on $X$. Show that $\deg(f) = 0$.

13.3. **Abel theorem.** The Abel map is defined for divisors in a natural way

$$\mathcal{A}(\sum n_i P_i) = \sum_i n_i \int_{P_0}^{P_i} \vec{\omega}.$$

If a divisor $D$ is of degree 0, then $\mathcal{A}(D)$ is independent of the base point $P_0$.

**Theorem 13.11.** *(Abel) The divisor $D \in \mathrm{Div}(X)$ is principal if and only if the following conditions hold:*

(1) $\deg D = 0$
(2) $\mathcal{A}(D) = 0$.

**References:**

J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.

## 14. Proof of Abel theorem

**Lemma 14.1.** *Let $X$ be a compact connected Riemann surface and let $P, Q \in X$ be two distinct points. Then there exists a meromorphic 1-form $\Omega$ such that*

1. *the only poles of $\Omega$ are at the points $P$ and $Q$ and $\mathrm{ord}_P(\Omega) = \mathrm{ord}_Q(\Omega) = -1$;*
2. $\mathrm{res}_P(\Omega) = -\mathrm{res}_Q(\Omega) = 1$.

*Proof.* Let $z_P$ is a local coordinate at $P$ with $z_P(P) = 0$ and $z_Q$ is a local coordinate at $Q$ with $z_Q(Q) = 0$. Fix a positive number $\epsilon$ small enough. Set

$$\begin{aligned} U'_P :=& \{x \mid z_P(x) < 2\epsilon\} \\ U_P :=& \{x \mid z_P(x) < \epsilon\} \\ U'_Q :=& \{x \mid z_Q(x) < 2\epsilon\} \\ U_Q :=& \{x \mid z_Q(x) < \epsilon\}. \end{aligned}$$

We choose $\epsilon$ so that $U'_P \cap U'_Q = \emptyset$. There exists a $C^\infty$-function $F : X \setminus \{P, Q\} \to \mathbb{C}/2\pi\mathbb{Z}$ such that

$$F(x) = \begin{cases} \log z_P(x) & , x \in U_P \\ -\log z_Q(x) & , x \in U_Q \\ 0 & , x \in X \setminus (U'_P \cup U'_Q) \end{cases}.$$

Consider a 1-form $\alpha := dF$. Note, that $\alpha$ is a $C^\infty$ 1-form on $X \setminus \{P, Q\}$. We observe that $d\alpha = 0$ and $d * \alpha$ is a $C^\infty$ 2-form on $X$. Indeed, $d * \alpha$ is $C^\infty$-smooth on $X \setminus \{U_P, U_Q\}$. Moreover, on $U_P \setminus \{P\}$

$$\alpha = \frac{dz_P}{z_P} \qquad \text{and} \qquad d * \alpha = d\left(i\frac{dz_P}{z_P}\right) = 0.$$

Analogously, $d * \alpha$ also vanishes on $U_Q \setminus \{Q\}$. We have

$$\begin{aligned} \int_X d * \alpha = \int_{X\setminus(U_P \cup U_Q)} d * \alpha = \int_{\partial(X\setminus(U_P\cup U_Q))} *\alpha \\ \int_{\partial U_P + \partial U_Q} -i\alpha = -i\int_{\partial U_P} \frac{dz_P}{z_P} + i\int_{\partial U_Q} \frac{dz_Q}{z_Q} = 0. \end{aligned}$$

Therefore, by the "Main theorem of compact Riemann surfaces" there exists $g \in C^\infty(X)$ such that

$$d * dg = d * \alpha \tag{5}$$

Set $\omega := \alpha - dg$. We have

$$d\omega = d\alpha - ddg = 0$$

and by (5)

$$d * \omega = d * \alpha - d * dg = 0.$$

Therefore, $\omega$ is a harmonic 1-form on $X\{P, Q\}$. Set $\Omega = \omega^{(1,0)}$. By Exercise 25 form $\Omega$ is a holomorphic 1-form on $X \setminus \{P, Q\}$. Moreover, $\Omega - \frac{dz_P}{z_P}$ is holomorphic in a neighborhood of $P$ and $\Omega + \frac{dz_Q}{z_Q}$ is holomorphic in a neighborhood of $Q$. This finishes the proof. □

**Lemma 14.2.** *Let $X$ be a compact connected Riemann surface, $P \in X$ a point and $z_P$ a local parameter around $P$ with $z_P(P) = 0$, and $n \in \mathbb{Z}_{\geq 2}$. Then there exists a meromorphic 1-form $\Omega$ such that*

1. *the only pole of $\Omega$ is at the point $P$ ;*

(2) $\mathrm{ord}_P(\Omega) = -n$;
(3) $\Omega = (\frac{1}{z_P^n} + O(1))dz_P$ *around* $P$.

These lemmas have a number of important corollaries.

**Corollary 14.3.** *Let* $X$ *be a compact connected Riemann surface. Then there exists a non-constant meromorphic function* $f : X \to \mathbb{CP}^1$.

**Corollary 14.4.** *Let* $X$ *be a compact connected Riemann surface of genus* 0*. Then* $X$ *is isomorphic to* $\mathbb{CP}^1$.

**Definition 14.5.** Let $P, Q \in X$ be two distinct points on a compact connected Riemann surface $X$ of genus $g$. Suppose that $(a_j, b_j)_{j=1}^g$ is a collection of closed cycles on $X\{P,Q\}$ forming a canonical basis of $H_1(X)$. We denote by $\Omega_{P,Q}$ the unique meromorphic 1-form such that:

(1) $(\Omega) = P - Q$;
(2) $\mathrm{res}_P(\Omega) = -\mathrm{res}_Q(\Omega) = 1$;
(3) $\int_{a_j} \Omega_{P,Q} = 0$ for $j = 1, \dots, g$.

**Exercise 28.** Prove that such a meromorphic 1-form exists and is unique.

**Lemma 14.6.** *Let* $\Omega_{P,Q}$ *be as above. Then*

$$\int_{b_j} \Omega_{P,Q} = \int_P^Q \omega_j, \qquad j = 1, \dots g.$$

*Proof of Abel theorem.*

**First** we show that if a divisor $D$ satisfies $\deg(D) = 0$ and $\mathcal{A}(D) = 0$ then $D$ is principal. Suppose that a divisor $D$ satisfies conditions (1) and (2) of Theorem 13.11. Since $D$ has degree 0 we can write

$$D = \sum_{i=1}^N P_i - \sum_{i=1}^N Q_i$$

for some points $P_1, \dots, P_N, Q_1, \dots Q_N$ on $X$ such that $P_i \neq Q_i$. Define

$$\Omega := \sum_{i=1}^N \Omega_{P_i Q_i}.$$

This meromorphic 1-form satisfies

$$\int_{a_j} \Omega = 0, \qquad j = 1, \dots g$$

and

$$\mathrm{res}_R \Omega = \mathrm{ord}_R(D) \text{ for all } R \in X.$$

By Lemma 14.6 we have

$$\int_{b_j} \Omega = \sum_{i=1}^N \int_{P_i}^{Q_i} \omega_j, \qquad j = 1, \dots, g.$$

This identities can be written in the vector form as

$$\int_{\vec{b}} \Omega = \sum_{i=1}^N \int_{P_i}^{Q_i} \vec{\omega},$$

where

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} \in H_1(X)^g \qquad \text{and} \qquad \vec{\omega} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \Omega^1_{\text{hol}}(X)^g.$$

By the assumption of the theorem the sum $\sum_{i=1}^N \int_{P_i}^{Q_i} \vec{\omega}$ belongs to the period lattice $2\pi i\mathbb{Z}^g + 2\pi i\mathcal{P}\mathbb{Z}^g$. Suppose that

$$\int_{\vec{b}} \Omega = 2\pi i\vec{m} + 2\pi i\mathcal{P}\vec{n} \qquad \text{for } \vec{m}, \vec{n} \in \mathbb{Z}^g.$$

For $\vec{k} \in \mathbb{Z}^g$ we have

$$\begin{aligned} \int_{\vec{b}} \vec{\omega}\,\vec{k} &= \int_{\vec{b}} (k_1\omega_1 + \ldots + k_g\omega_g) = \\ &= 2\pi i \left( \sum_{i=1}^g k_i B_{ji} \right)_{j=1}^g = 2\pi i\mathcal{P}\,\vec{k}. \end{aligned}$$

We set

$$\widetilde{\Omega} = \Omega - \vec{\omega}^t \cdot \vec{n}.$$

Then

$$\int_{\vec{b}} \widetilde{\Omega} = 2\pi i\vec{m} + 2\pi i\mathcal{P}\vec{n} - 2\pi i\mathcal{P}\vec{n} = 2\pi i\vec{m} \tag{6}$$

$$\int_{\vec{a}} \widetilde{\Omega} = \int_{\vec{a}} (\Omega - \vec{\omega}^t \cdot \vec{n}) = 0 - 2\pi i\vec{n} = -2\pi i\vec{n}. \tag{7}$$

Also for each point $R \in X$ we have

$$\operatorname{res}_R(\widetilde{\Omega}) = \operatorname{ord}_R(D) \in \mathbb{Z}. \tag{8}$$

We define the function $\widetilde{f} : X \setminus \operatorname{supp}(D) \to \mathbb{C}/2\pi i\mathbb{Z}$

$$\widetilde{f}(P) = \int_{P_0}^{P} \widetilde{\Omega}.$$

The equations (6)–(8) imply that function $\widetilde{f}$ is well defined. Set

$$g := \exp(\widetilde{f}).$$

Now an easy computation shows that $g$ is a meromorphic function and

$$(g) = \sum_{R \in X} \operatorname{ord}_R(D) = D.$$

Hence divisor $D$ is principal.

**Next** we show that any principal divisor $D$ satisfies conditions (1) and (2) of Theorem 13.11. Let $g$ be a meromorphic function on $X$ and $D = (g)$. Exercise 27 implies that $\deg(D) = 0$. Now it suffices to prove that $\mathcal{A}(D) = 0$. Set

$$\alpha = \frac{dg}{g} = d(\log(g)).$$

Note that $\alpha$ is a meromorphic differential form on $X$ while $\log(g)$ is defined only locally on $X \setminus \operatorname{supp}(D)$. Moreover, for each $P \in \operatorname{supp}(D)$ we have

$$\operatorname{ord}_P(\alpha) = -1 \qquad \text{and} \qquad \operatorname{res}_P(\alpha) = \operatorname{ord}_P(g).$$

We have

$$0 = \int_{X \setminus \mathrm{supp}(D)} \alpha \wedge \omega_j.$$

Let $\Pi$ be a simply connected polygonal model of $X$. We suppose that $\{a_j, b_j\}_{j=1}^g$ are the sides of $\Pi$. Fix $P_0 \in \Pi \setminus \mathrm{supp}(D)$. Set

$$F_j(P) := \int_{P_0}^P \omega_j.$$

Note that $F_j$ is well defined on $\Pi$ (not on $X$) and $\omega_j = dF_j$. We have

$$0 = \int_{\Pi \setminus \mathrm{supp}(D)} \omega_j \wedge \alpha = \int_{\partial(\Pi \setminus \mathrm{supp}(D))} F_j \alpha \tag{9}$$

We can write

$$\int_{\partial(\Pi \setminus \mathrm{supp}(D))} F_j \alpha = \int_{\partial \Pi} F_j \alpha - 2\pi i \sum_{P \in \mathrm{supp}(D)} \mathrm{res}_P(F_j \alpha) \tag{10}$$

Note that

$$\begin{aligned} \int_{\partial \Pi} F_j \alpha &= \sum_{i=1}^g \left( \int_{a_i} \omega_j \int_{b_i} \alpha - \int_{a_i} \alpha \int_{b_i} \omega_j \right) \\ &= 2\pi i \int_{b_j} \alpha - 2\pi i \sum_{i=1}^g B_{ij} \int_{a_i} \alpha. \end{aligned}$$

Since $\alpha = d(\log(g))$, all periods of $\alpha$ (integrals over closed cycles in $X \setminus \mathrm{supp}(D)$) are in $2\pi i \mathbb{Z}$. Suppose that

$$\int_{\vec{a}} \alpha = 2\pi i \vec{m} \qquad \text{and} \qquad \int_{\vec{b}} \alpha = 2\pi i \vec{n}$$

for some $\vec{m}, \vec{n} \in \mathbb{Z}^g$. Then we have

$$\left( \int_{\partial \Pi} F_j \alpha \right)_{j=1}^g = -4\pi^2 (\vec{m} - \mathcal{P} \vec{n}) \in 2\pi i \Lambda. \tag{11}$$

Here $\Lambda$ benotes the period matrix of $X$. On the other hand, since $F_j$ is holomorphic at each $P \in$ supp(D) and $\alpha$ has a pole of order 1 and residue $\mathrm{ord}_P(g)$, we arrive at

$$\mathrm{res}_P(F_j \alpha) = F_j(P) \mathrm{ord}_P(g) = \mathrm{ord}_P(g) \int_{P_0}^P \omega_j.$$

Combining these identities for all $j = 1, \ldots, g$ and $P \in \mathrm{supp}(D)$ we obtain

$$\left( \sum_{P \in \mathrm{supp}(D)} \mathrm{res}_P(F_j \alpha) \right)_{j=1}^g = \sum_{P \in \mathrm{supp}(D)} \mathrm{ord}_P(g) \int_{P_0}^P \vec{\omega} = \mathcal{A}(D) \bmod \Lambda. \tag{12}$$

Now the identities (9)–(12) imply that

$$\mathcal{A}(D) = 0.$$

This finishes the proof. $\square$

## 15. Riemann-Roch theorem

**Definition 15.1.** Let $D$ be a divisor on $X$. We consider the vector spaces

$$\mathcal{L}(D) = \{f \text{ meromorpic functions on } X \mid (f) \geq D \text{ or } f \equiv 0\}$$

and

$$\mathcal{H}(D) = \{\Omega \text{ meromorpic differentials on } X \mid (\Omega) \geq D \text{ or } \Omega \equiv 0\}.$$

Set

$$\ell(D) := \dim_{\mathbb{C}}\mathcal{L}(D) \text{ and } i(D) := \dim_{\mathbb{C}}\mathcal{H}(D).$$

**Remarks**

(1) $D_1 \geq D_2$ implies $\mathcal{L}(D_1) \subseteq \mathcal{L}(D_2)$ and $\ell(D_1) \leq \ell(D_2)$.
(2) $\mathcal{L}(0) = \{$ constants $\}$ and $\ell(0) = 1$.
(3) $\deg(D) \geq 0$. $D \neq 0$ implies $\ell(D) = 0$.
(4) $H(0) = \{$ space of holomorphic differentials $\}$ and $i(0) = g$.

**Lemma 15.2.** *$\ell(D)$ and $i(D)$ depend only on the divisor class of $D$ and $i(D) = \ell(D - C)$ where $C$ is the canonical divisor class.*

**Theorem 15.3.** *(Riemann-Roch) Let $X$ be a compact Riemann surface of genus $g$ and $D$ be a divisor on $X$. Then*

$$\ell(-D) = \deg(D) - g + 1 + i(D).$$

**Corollary 15.4.** *Let $X$ be a compact Riemann surface of genus 0. Then $X$ is conformally equivalent to the complex projective line $\mathbb{PC}^1$.*

**Corollary 15.5.** *Let $X$ be a compact Riemann surface of genus $g \geq 1$. Then there is no $z \in X$ at which all holomorphic 1-forms vanish.*

**References:**

J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.

## References

[1] J. B. Bost, Introduction to Compact Riemann Surfaces, Jacobians, and Abelian Varieties, Springer-Verlag, 1992. Lectures given at the meeting "Number Theory and Physics", Les Houches, March 7-16, 1989

[2] S. Donaldson, Riemann Surfaces, Oxford University Press, 2011.

[3] J. Jost, Compact Riemann Surfaces, An Introduction to Contemporary Mathematics, Third Edition, Springer-Verlag Berlin Heidelberg, 2006.